LECTURE 14

11.03.2024

Classical Propositional Logic

Alphabet: p, $|$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $($, $)$

We will use the following abbreviation:

p followed by n $|$'s is denoted by p_n.

Let $\wp$ denote a countable set of these propositional variables p_n's.

Language ($\mathcal{L}$):

$$\varphi, \psi ::= p_n \mid \neg\varphi \mid \varphi\wedge\psi \mid \varphi\vee\psi \mid \varphi\rightarrow\psi \mid \varphi\leftrightarrow\psi,$$

where $p_n \in \wp$.

Example: 2 is even : φ
2 is prime : ψ

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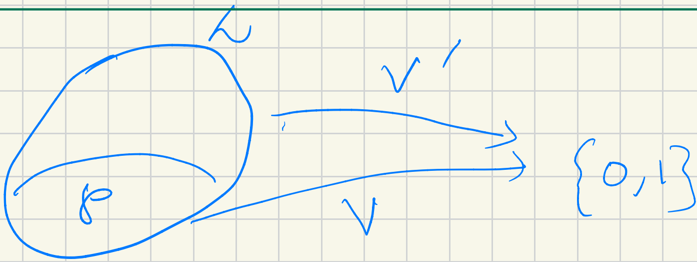
2 is even and prime : $\varphi\wedge\psi$

Model:

$V : \mathcal{P} \to \{0,1\}$, a function from the set of propositional variables to $\{0,1\}$.
We term this V as a 'valuation'.

Any $V : \mathcal{P} \to \{0,1\}$ can be uniquely extended to $V' : \mathcal{L} \to \{0,1\}$.



Truth tables:

(1) for $\neg$:

	$\neg$
0	1
1	0

(2) for $\wedge$:

$\wedge$	0	1
0	0	0
1	0	1

(3) for $\vee$:

$\vee$	0	1
0	0	1
1	1	1

(4) for $\to$:

$\to$	0	1
0	1	1
1	0	1

(5) for $\leftrightarrow$:

$\leftrightarrow$	0	1
0	1	0
1	0	1

Example:

Consider a valuation V, s.t.

$V(p_1) = 1$, $V(p_2) = 0$, $V(p_3) = 1$.

Then, $V'((p_1 \vee p_3) \to p_2) = 0$

$$V'((\neg p_1 \vee p_3) \to p_2) = 0$$

$$V'((p_1 \to p_3) \to p_2) = 0$$

Since V can be uniquely extended to V', we will denote V' by V.

Let Γ be a set of formulas and φ be a formula in classical propositional logic. We define $\Gamma \vDash \varphi$ (semantic consequence relation) and $\Gamma \vdash \varphi$ (deductive consequence relation) as earlier.

Soundness theorem.

If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.

We have already discussed the proof procedure for soundness theorem.

So, once we show that the axioms are valid and the rules preserve consequence, we are done.

But, we do not know what these axioms and rules are. We now focus on the proof of completeness theorem to find them.

Let's find some valid formulas and some rules that preserve consequence.

- $\varphi \to \varphi$
- $(\varphi \wedge \psi) \to \varphi$
- $\varphi \to (\varphi \vee \psi)$

$\Big\}$ valid formulas

$$\frac{\varphi \quad \varphi \to \psi}{\psi} \qquad \frac{\varphi \vee \psi \quad \neg\varphi \vee \chi}{\psi \vee \chi}$$ } rules preserving consequence

Completeness Theorem

If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

How to prove the theorem?

We explore some properties of $\models$ and try to see whether we can prove such properties for $\vdash$.

Property 1: If $\Gamma \models \varphi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \models \varphi$.

We want to show: If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \varphi$: It follows from the definition of $\vdash$.

Property 2: $\{\varphi_1, \varphi_2, \ldots, \varphi_n\} \models \psi$

iff

$\models (\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n) \to \psi$.

In other words we have :

$$\Gamma \cup \{\varphi\} \models \psi \text{ iff } \Gamma \models \varphi \to \psi .$$

Let us now try to show :

$$\Gamma \cup \{\varphi\} \vdash \psi \text{ iff } \Gamma \vdash \varphi \to \psi .$$

(Deduction Theorem)

Proof : (if-part) Suppose $\Gamma \vdash \varphi \to \psi$.

To prove that $\Gamma \cup \{\varphi\} \vdash \psi$.

Now, $\Gamma \cup \{\varphi\} \vdash$ 1. φ (Premise)

2. $\varphi \to \psi$ (Assumption)

3. ψ (by M.P. on 1,2)

[Let us consider the rule $\boxed{\dfrac{\varphi \quad \varphi \to \psi}{\psi}}$

which we know as Modus Ponens (M.P.)]

This completes the proof of if-part.

(only-if part) Let $\Gamma \cup \{\varphi\} \vdash \psi$.

We have to show that $\Gamma \vdash \varphi \to \psi$.

prove this by applying induction on the length of a derivation/proof of ψ from $\Gamma \cup \{\varphi\}$.

Base Case : $n = 1$. Then ψ is either a member of $\Gamma \cup \{\varphi\}$ or ψ is an axiom.

Case 1 . $\psi = \varphi$

Then, we have to show $\Gamma \vdash \varphi \to \varphi$.

[Let us consider an axiom $\boxed{\varphi \to \varphi}$]

We have: $\Gamma \vdash \varphi \to \varphi$.

Case 2 $\psi \in \Gamma$

We have to show $\Gamma \vdash \varphi \to \psi$

Since $\psi \in \Gamma$, we have $\Gamma \vdash \psi$.

[Let us consider an axiom

$\boxed{\psi \to (\varphi \to \psi)}$]

Then, we have the following

derivation of $\varphi \to \psi$ from Γ :

$\Gamma \vdash$ 1. ψ (Premise)

2. $\psi \to (\varphi \to \psi)$ (Axiom)

3. $\varphi \to \psi$ (M.P. on 1, 2).

Case 3. ψ is an axiom.

To show $\Gamma \vdash \varphi \to \psi$.

We have $\Gamma \vdash$ 1. ψ (Axiom)

2. $\psi \to (\varphi \to \psi)$ (Axiom)

3. $\varphi \to \psi$ (M.P. on 1, 2)

We are done with the bases cases.

So, to prove the base cases we consider

axioms: (1) $\varphi \to \varphi$

(2) $\varphi \to (\psi \to \varphi)$, and

rule: $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$ (M.P.).