## Lecture 14

As we mentioned in the last class, we have a model set \$ , say . We need to find a model Ms = (Ds, Is, Ys) s.t. Ms Fq M qEA. Now, suppose that we have Ms (by magic, ray). To prove Ms Fq iff QEK, we apply induction on the size of q Base case : (1)  $\varphi := t_1 \equiv t_2$ (2)  $\varphi := p_1^n t_1 \cdots t_n$ Induction Mypollusis : Suppose the result holds for all formulas of size < m Induction Step: Consider q to be of size m+1. Then we have the following cases : (1)  $q := \chi q$  (2)  $q := \chi V \chi, \chi \Lambda \chi, \chi \to \chi, \chi \to \chi$ (3) q := Vr4, Jr4 Let us now suppose that the result for the

base cases hold fet us now consider the induction step cases (1) q := 74. [MD FQ iff QEA] MaFq iff MaF7Y iff Matty if YED H JYED H QED. (2) q != YVX [Matq iff QEA] Moto ill Motovill Moto n Moto HYEDOXED HEAVYED HEA (Sinicharly, the proofs for the cases YAX, Y-JX, YK-)X) For proving we need the CPL anions and rule Arion and rule set (1)  $A1 q \rightarrow (\gamma \rightarrow q)$  $A2 \left( \varphi \rightarrow (\psi \rightarrow \chi) \right) \rightarrow \left( \left( \varphi \rightarrow \psi \right) \rightarrow (\varphi \rightarrow \chi) \right)$  $A3. (2q \rightarrow 4) \rightarrow ((2q \rightarrow 24) \rightarrow q)$  $R_{1} \xrightarrow{\varphi} \varphi \rightarrow \psi$ We still need to show the base cases and Induction Step (Case 3).

Let us move on to the base cases  $q' = t_1 \equiv t_2$ ,  $q' = p_1'' t_1 \cdots t_n$ . what is Ms, i.e., what are Ds, Is, ys? het us de fine DA = {t: t is a term? Define Is as follows !  $I_{A}(c) = c$  for all  $c \in C$  $T_{\Delta}(f_{i}^{n}): D_{\Delta}^{n} \rightarrow D_{A}: f_{i}^{n}$  $I_{\Delta}(p_{i}) \subseteq D_{\Delta}^{n}: (t_{i}, \dots, t_{n}) \in I_{\Delta}(p_{i})$  if  $p_{i}t_{i} \dots t_{n} \in \Delta$ Define  $y_A: V \to D_A: y_A(x) = x$ Now, we need to estend ys to get the values for all terms in the language De have . Proposition 'fa(t) = t for all terms t. H.W. Prove this proposition. So, we have Mo = (DA, JA, YA) We need to show Ms Fq iff QEA.

Base Cases (1)  $t_1 \equiv t_2$ (2)  $p_i^n$   $t_i = t_n$ , het us first try to prove (2). We want to prove :  $M_{\Delta} \neq p_i^n t_1 - . t_n \quad \text{iff} \quad p_i^n t_i - . t_n \in \Delta$ We have : Mot pit, --. tr  $\mathcal{H}\left(\mathcal{Y}_{\Delta}(t_{1}), \cdots, \mathcal{Y}_{\Delta}(t_{n})\right) \in \mathcal{I}_{\Delta}(p_{1}^{n})$  $\mathcal{M} \quad (t_1, \dots, t_n) \in \mathcal{I}_{\mathcal{S}} \left( p_i^{\mathsf{T}} \right)$  $i \not = p_i^n t_i - - t_n \in \Delta$ This completes the proof of (2). Let us now move on to Bese case (1). To prove :  $M_{\Delta} \neq t_1 \equiv t_2$  iff  $t_1 \equiv t_2 \in \Delta$ . Now,  $M_{\Delta} \neq t_1 \equiv t_2$  iff  $l_{f_{\Delta}}(t_1) = l_{f_{\Delta}}(t_2)$ iff  $t_1 = t_2 = (=t, say) *$ H  $t_1 = t_2 \in \Delta$  ?? ( im language of arithmetic

★ 1=1, 2=2, ~ , but there is no way to equate 2+1 and 3) Let's go back to our domain of definition We had  $D_{\delta} = \{t : t \text{ is a lerm}\}$ . ~ is an equivalence relation. Proof. (1) We have to show that ~ is reflexive, i.e.,  $t = t \in \Delta$  for all terms t Aniom ! t = t , for all terms t ] ble shave I t = t , and hence  $f \equiv t \in \Delta$ . Cas f q implies  $\Delta f q$  implies QED, as D is consistent and complete) (2) Anion !  $(t_1 \equiv t_2) \rightarrow (t_2 \equiv t_1)$ (3) Aniom:  $(t_1 \equiv t_2) \rightarrow ((t_2 \equiv t_3) \rightarrow (t_1 \equiv t_3))$ So, we have ~ is an equivalence relation. We can define the equivalence

classes, and we have a new domann,  $D_{\mathcal{S}} = \{ [t] : t \text{ is a term } \}, \text{ where } [t] \text{ densities}$ the equivalence class of t. Now, we have to define Is as follows !  $\left[ \int_{A} (e) - [e] \right]$  $\overline{I}_{\Delta} \left( f_{i}^{n} \right) : D_{\Delta}^{n} \rightarrow D_{\lambda}^{n} : \overline{I}_{\Delta} \left( f_{i}^{n} \right) \left( [t_{i}], \dots, [t_{n}] \right)$  $= \left[ f_{i}^{n} t_{i} - \cdots t_{n} \right]$  $I_{\lambda}(p_{i}^{n}) \subseteq D_{0}^{\prime n}$ ; (Et, ], ..., Etn])  $\in I_{\lambda}(p_{i}^{n})$ place, we need to check that I's(f") and Is (pr) are well defined. - I (f") is well - de fine d.  $= f't' - t_h \in \Delta$ 

 $\left[ Aniom \left( t_i = t_i' \land - \cdot \land t_n = t_n' \right) \rightarrow \left( f_i' t_i - t_n = f_i' t_i' - t_n' \right) \right]$ With this assiron we have our well-definedness of  $I_{\Delta}(f_i^n)$ , as before. - I (p) is well-defined. H.W. Prove the state ment above. Finally, let us define lys: V -> D's as follows: ly's(n) = [n] As earlier we can show the following : froposition:  $f_A(t) = [t]$  for all terms  $t \in J$ . H.W. Prove the proposition het us go back to the proof of the main result; Mate iff 'qED, for all q. Bare case: O t\_=t\_: Mot t\_=t\_ iff 

 $\mathcal{M}\left(\mathcal{Y}_{\mathcal{S}}(\mathsf{t}_{i}), \cdots, \mathcal{Y}_{\mathcal{S}}(\mathsf{t}_{n})\right) \in \mathcal{I}_{\mathcal{S}}\left(\mathsf{p}_{i}^{n}\right) \quad \mathcal{M}$  $([t_i], \dots, [t_n]) \in I_A(p_i^n)$  iff  $p_i^n t_i \dots t_n \in A$ So, we are done with the base cases Now, by our previous arguments, we are also done for the formulas 74,  $\forall \forall X$ ,  $\forall A X$ ,  $\forall \neg X$ ,  $\forall \leftrightarrow X$ . Thus, if we only consider atomic formulas and their Boolean combinations, which we term as , zeroth order logic , then we have the completines theorem for this logie. Now, let us consider the quantified formular, Yny and Iny We will be done if we can show My F VAY if VAYED Ma F J ay H J ay E A