Lecture 15

Before moving forward with the quantified formula case, let us recapitulate the axioms we have used till now.

Axiom and rule set (1)

A1. $\varphi \to (\psi \to \varphi)$

A2. $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

A3. $(\neg\varphi \to \psi) \to ((\neg\varphi \to \neg\psi) \to \varphi)$

R1. $$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$

Axiom and rule set (2) [Equality axioms]

A4. $t \equiv t$

A5. $(t_1 \equiv t_2) \to (t_2 \equiv t_1)$

A6. $(t_1 \equiv t_2) \to ((t_2 \equiv t_3) \to (t_1 \equiv t_3))$

A7 $((t_1 \equiv t_1') \wedge \cdots \wedge (t_n \equiv t_n')) \to (f_i^n t_1 \cdots t_n \equiv f_i^n t_1' \cdots t_n')$

A8 $((t_1 \equiv t_1') \wedge \cdots \wedge (t_n \equiv t_n')) \to (p_i^n t_1 \cdots t_n \leftrightarrow p_i^n t_1' \cdots t_n')$

Rather than having all these equality axioms we can consider the following:

Axiom and rule set (2) [Alternate]

(A4') $t \equiv t$

(A5') $(t \equiv t') \to (\varphi \leftrightarrow \varphi')$, where φ' is obtained by replacing some occurrences of t by t' and some occurrences of t' by t

H.W. Use (A4') and (A5') to derive (A4) - (A8).

Let us now get back to the proof of the main result. We have to show:

$$\mathcal{M}_\Delta' \models \forall x \psi \text{ iff } \forall x \psi \in \Delta.$$

Due to the construction of $\mathcal{M}_\Delta'$, it is enough to show that:

(a) if $\forall x \varphi \in \Delta$, then $\varphi[t/x] \in \Delta$ for all terms t.

(b) if $\exists x \varphi \in \Delta$, then $\varphi[t/x] \in \Delta$ for some term t.

Towards getting (a), we consider the following:

Axiom: $\forall x \varphi \to \varphi[t/x]$, t is substitutable for x in φ.

Towards getting (b) we introduce the notion of witness-fulfilledness.

Witness-fulfilled set of formulas

A set of formulas Δ is said to be witness-fulfilled if for every formula of the form $\exists x \varphi$, if $\exists x \varphi \in \Delta$, then $\varphi[t/x] \in \Delta$ for some term t.

Proposition: Any consistent, complete set of formulas Γ can be extended to a consistent, complete, witness-fulfilled set of formulas.

Proof: We first note that a constant symbol c not occuring in φ can always be substitutable for x in φ. We use this idea below. We expand our language $\mathcal{L}$ with a countable

set of new constant symbols, $\mathcal{D}$, say,
where $\mathcal{D} = \{d_1, d_2, \ldots\}$. $\mathcal{L} : (\mathcal{C}, \mathcal{F}, \mathcal{P})$;
$\mathcal{L}' : (\mathcal{C} \cup \mathcal{D}, \mathcal{F}, \mathcal{P})$. Let us enumerate formulas
of $\mathcal{L}'$: $\beta_0, \beta_1, \beta_2, \ldots$

Let $\Delta_0 = \Gamma$ (consistent and complete).
(Lindenbaum construction)
We construct $\Delta_1, \Delta_2, \ldots, \Delta_k, \ldots$ as follows:

If $\beta_k = \exists x \varphi$, then

$$\Delta_{k+1} = \begin{cases} \Delta_k \cup \{\exists x \varphi, \varphi[d/x]\}, & \text{where } d \text{ is the least-indexed constant symbol not occuring in } \Delta_k \text{, and the whole set is consistent.} \\ \Delta_k, & \text{otherwise.} \end{cases}$$

else,

$$\Delta_{k+1} = \begin{cases} \Delta_k \cup \{\beta_k\}, & \text{if it is consistent} \\ \Delta_k, & \text{otherwise} \end{cases}$$

We have:

$\Gamma = \Delta_0 \subseteq \Delta_1 \subseteq \Delta_2 \subseteq \ldots$

Let $\Delta = \bigcup_{n \geq 0} \Delta_n$

Claim: Δ is consistent, complete and witness-fulfilled.

To prove the claim it is enough to show the following:

Proposition: Let Γ be a consistent and complete set of formulas and $\exists x \varphi \in \Gamma$. Let c be a constant symbol not occuring in Γ. Then, $\Gamma \cup \{\varphi[c/x]\}$ is consistent.

Proof: Suppose $\Gamma \cup \{\varphi[c/x]\}$ is inconsistent. Then, $\Gamma \vdash \neg\varphi[c/x]$. Since c does not occur in Γ, we have $\Gamma \vdash \forall x \neg\varphi$.

[Assume the following: if $\Gamma \vdash \psi[c/x]$, where c does not occur in Γ or ψ. Then, $\Gamma \vdash \forall x \psi$ (Generalisation of constants)]

Since $\Gamma \vdash \forall x \neg\varphi$, $\forall x \neg\varphi \in \Gamma$, a contradiction to the consistency of Γ, as $\exists x \varphi \in \Gamma$. Hence we have our result. This basically completes the proof of the proposition. ▨

H.W. Finish the proof in details, assuming the 'Generalisation of constants' result.

Let us now make an observation regarding 'adding witnesses'. To take care of this we used the idea of 'adding new constant symbols', which is famously known as the 'Gödel trick': expand the language L to L' with new constant symbols and add them as witnesses. We note that this makes us deal with the expanded language, and whatever proof we have henceforth will be in the expanded language. We could do this and later get back to the language we started with.

Observation: There is no need to expand the language. New variables just work as well as the constant symbols.

What do we do?

- The set of variables is given by $V = \{x_1, x_2, \ldots\}$.
- In the given consistent set Γ that we start with, replace every variable x_i by x_{2i}. Now, all the odd-indexed variables are unused now.

- We show that the new set of formulas thus formed, say Γ', is consistent as well.

- In the Lindenbaum construction, whenever we come across a formula of the form $\exists x \varphi$, we add $\varphi[y/x]$, where y is the least unused variable.

- We need to prove that $\varphi[y/x]$ can be added consistently: this can be shown by an analogous proposition to the one above.

What we discussed above suggests proving the following proposition, which is a version of the generalisation of the constants result.

- Let Γ be a consistent set of formulas, φ be a formula and y be a variable such that the variable y does not occur in Γ or φ. Then, if $\Gamma \vdash \varphi[y/x]$ then $\Gamma \vdash \forall x \varphi$.

To prove the result above, we first need to show the following:

Alphabetic-equivalence proposition: Let φ be a formula, x be a variable, t be a term. Then, there exists a formula φ' such that $\vdash \varphi \leftrightarrow \varphi'$ and t is substitutable for x in φ'.

We will continue with the proof in the next class.