LECTURE 16

18.03.2024

The axiom system for CPL:

Axioms (Schema)

1. $\varphi \to (\psi \to \varphi)$
2. $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
3. $(\neg\varphi \to \psi) \to ((\neg\varphi \to \neg\psi) \to \varphi)$

Rule (M.P.)

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$

Let us go back to the soundness proof before finishing this discussion. To prove soundness, we have to show that the axioms are valid, and the rule preserve consequence.

Validity of Axiom 1.

Axiom 1: $\varphi \to (\psi \to \varphi)$

To show that Axiom 1 is valid, we need to show that for all valuations V, $V(\varphi \to (\psi \to \varphi)) = 1$.

Suppose not. There exists a valuation V', say s.t. $V'(\varphi \to (\psi \to \varphi)) = 0$.

Then, $V'(\varphi) = 1$ and $V'(\psi \to \varphi) = 0$ which implies $V'(\psi) = 1$ and $V'(\varphi) = 0$.

So, we arrive at a contradiction.

Hence, $V(\varphi \to (\psi \to \varphi)) = 1$ for all valuations V. Thus $\varphi \to (\psi \to \varphi)$ is a valid formula.

H.W. Prove that Axiom 2 and Axiom 3 are valid.

Rule preserves consequence

$$\frac{\varphi \quad \varphi \to \psi}{\psi}$$

: We need to show that for any valuation V, if $V(\varphi) = 1$ and $V(\varphi \to \psi) = 1$, then $V(\varphi) = 1$. This follows.

Thus we finish the proof for soundness theorem: If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.

Thus: $\Gamma \vdash \varphi$ iff $\Gamma \models \varphi$ in CPL.

This theorem is known as the 'generalised completeness theorem' which we proved for CPL. Now, it follows that $\vdash \varphi$ iff $\models \varphi$.

$\vdash \varphi$ ↓ φ is a theorem in CPL

$\models \varphi$ ↓ φ is a validity or, a tautology in CPL.

This result is what we call 'completeness theorem'. There are logics where completeness theorem holds but not the generalised completeness theorem. And also, there are logics where completeness theorem does not hold.

H.W. Prove that the generalised completeness in

CPL implies compactness.

Theorems in CPL.

1. $\vdash \varphi \to \varphi$.

Proof. (a) $\{\varphi\} \vdash \varphi$.

$\vdash \varphi \to \varphi$ (by D.T.).

(b) 1. $\varphi \to ((\varphi \to \varphi) \to \varphi)$ Axiom 1.

2. $(\varphi \to ((\varphi \to \varphi) \to \varphi)) \to ((\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi))$ Axiom 2

3. $(\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)$ (M.P. 1,2)

4. $\varphi \to (\varphi \to \varphi)$ Axiom 1

5. $\varphi \to \varphi$ (M.P. 4,3).

2. $\vdash \neg\neg\varphi \to \varphi$

Proof. 1. $\neg\neg\varphi \to (\neg\varphi \to \neg\neg\varphi)$ Axiom 1

2. By D.T. we have $\{\neg\neg\varphi\} \vdash \neg\varphi \to \neg\neg\varphi$.

3. $(\neg\varphi \to \neg\varphi) \to ((\neg\varphi \to \neg\neg\varphi) \to \varphi)$ Axiom 3

4. $\neg\varphi \to \neg\varphi$ Theorem

5. $(\neg\varphi \to \neg\neg\varphi) \to \varphi$ (M.P. 4,3)

6. φ (M. P. 2, 5)

7. $\neg\neg\varphi \to \varphi$ D. T.

Deductive Consequence relation in FOL.

Axioms

1. All propositional tautologies.
2. $t = t$ for all terms t.
3. $(t = t') \to (\varphi \Leftrightarrow \varphi')$, where φ' is obtained from φ by replacing some occurrences of t by t' and some occurences of t' by t.
4. $\forall x \varphi \to \varphi[t/x]$, for any term t substitutable for x in φ.

5. $\varphi \to \forall x\, \varphi$, for $x \notin FV(\varphi)$

6. $\forall x\, (\varphi \to \psi) \to (\forall x\, \varphi \to \forall x\, \psi)$

Rules

$$\frac{\varphi \quad \varphi \to \psi}{\psi} \quad \text{(M.P.)} \qquad \qquad \frac{\vdash \varphi}{\vdash \forall x\, \varphi} \quad \text{(Generalization)}$$

They form the Axiom System for FOL.

H.W. Prove that the axioms are valid and the rules preserve consequence.

- Let Γ be a set of formulas and φ be a formula. φ is said to be a deductive consequence of Γ ($\Gamma \vdash \varphi$) if there is a sequence of formulas $\varphi_1, \varphi_2, \dots, \varphi_n$, such that $\varphi_n = \varphi$ and each φ_i is either

  - a member of Γ, or

- an axiom, or
- obtained by some rule.

We have defined the semantic consequence relation ($\Gamma \models \varphi$) earlier.

Soundness theorem.

If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.

Completeness theorem.

If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

Some theorems in FOL

1. $\vdash Px \to \exists y\, Py$

(a) $(\forall y \neg Py \to \neg Px) \to (Px \to \neg \forall y \neg Py)$ Ax 1.

[$(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$ is a validity / tautology in CPL]

(b) $\forall y \neg Py \to \neg Px$ Ax. 4

(c) $Px \to \neg \forall y \neg Py$ (M.P. (b), (a))

(d) $Px \to \exists y Py$

2. $\vdash \exists x \forall y \varphi \to \forall y \exists x \varphi$.

It is enough to show that:

$$\{\exists x \forall y \varphi\} \vdash \forall y \exists x \varphi. \quad \text{(by D.T.)}$$

[<u>Generalisation Theorem</u>: If $\Gamma \vdash \varphi$ and x does not occur free in Γ, then $\Gamma \vdash \forall x \varphi$]

Using this theorem we can say that it is enough to prove $\{\exists x \forall y \varphi\} \vdash \exists x \varphi$.

This is same as showing:

$$\{\neg \forall x \neg \forall y \varphi\} \vdash \neg \forall x \neg \varphi.$$

So, it is enough to show:

$$\{\forall x \neg \varphi\} \vdash \forall x \neg \forall y \varphi.$$

So, it is enough to show:

$$\{\forall x \neg \varphi\} \vdash \neg \forall y \varphi$$ (by the Generalisation Theorem)

Then, it is enough to show

$$\{\forall x \neg \varphi, \forall y \varphi\} \vdash \bot$$ (bottom)

that is, $\{\forall x \neg \varphi, \forall y \varphi\} \vdash \psi \wedge \neg \psi$ for some formula ψ.

Let us now prove this.

1. $\forall x \neg \varphi \to \neg \varphi[x/x]$ Ax. 4.
2. $\forall x \neg \varphi$ Premise
3. $\neg \varphi$ (M.P. 2, 1)
4. $\forall y \varphi \to \varphi[y/y]$ Ax 4.
5. $\forall y \varphi$ Premise
6. φ (M.P. 5, 4)
7. $\varphi \wedge \neg \varphi$ (($\psi \to (\chi \to (\psi \wedge \chi))$ is a prop. taut.)

This basically shows :

$$\{\exists x \, \forall y \, \varphi\} \vdash \forall y \, \exists x \, \varphi .$$

So, $\vdash \exists x \, \forall y \, \varphi \rightarrow \forall y \, \exists x \, \varphi$.

H. W. :

In CPL :

(1) $\varphi \rightarrow \neg\neg\varphi$.

(2) $\neg\varphi \rightarrow (\varphi \rightarrow \psi)$

(3) $\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$

In FOL :

(1) $\exists x (\varphi \wedge \psi) \rightarrow (\exists x \, \varphi \wedge \exists x \, \psi)$

(2) $(\forall x \, \varphi \vee \forall x \, \psi) \rightarrow \forall x (\varphi \vee \psi)$

H. W. Give the proof of the Generalisation theorem.