MODAL LOGIC

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THE HISTORY OF MODAL LOGIC

- R. Goldblatt. *Mathematical Modal Logic: A view of its evolution*. Handbook of the History of Logic, Vol. 7, 2006.
- P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Section 1.7, Cambridge University Press, 2001.
- R. Ballarin. *Modern origins of Modal Logic*. Stanford Encyclopedia of Philosophy, 2010.

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TYPES OF MODAL LOGIC

- **temporal**: now, tomorrow, yesterday, since, until, eventually, henceforth
- **epistemic**: knowledge, general knowledge, common knowledge
- **doxastic**: belief, general belief, common belief
- **preference**: preferred, strictly preferred, equally preferred
- **deontic**: obligatory, forbidden, permitted
- **dynamic**: after some action, program, computation
- **spatial**: locally, beyond
- **metalogic**: validity, satisfiability, provability, consistency

BASIC MODAL LANGUAGE

- Any propositional variable is a formula
- If P and Q are formulas, then so are $\neg P$, $P \wedge Q$, $P \vee Q$, $P \rightarrow Q$
- If P is a formula then so are $\Box P$ and $\Diamond P$

BASIC MODAL LANGUAGE

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Examples:

$\Diamond(\Box P \rightarrow Q) \wedge \neg \Diamond R$, $\Box \bot$, $\Diamond \Box \top$

SEMANTICS

- Relational Semantics (Kripke)
- Algebraic Semantics (Boolean algebra with operators)
- Neighbourhood Semantics (Lifting of relations)
- Topological Semantics (Closure and interior operators)
- Category-theoretic Semantics (Coalgebra)

KRIPKE SEMANTICS

The main idea:

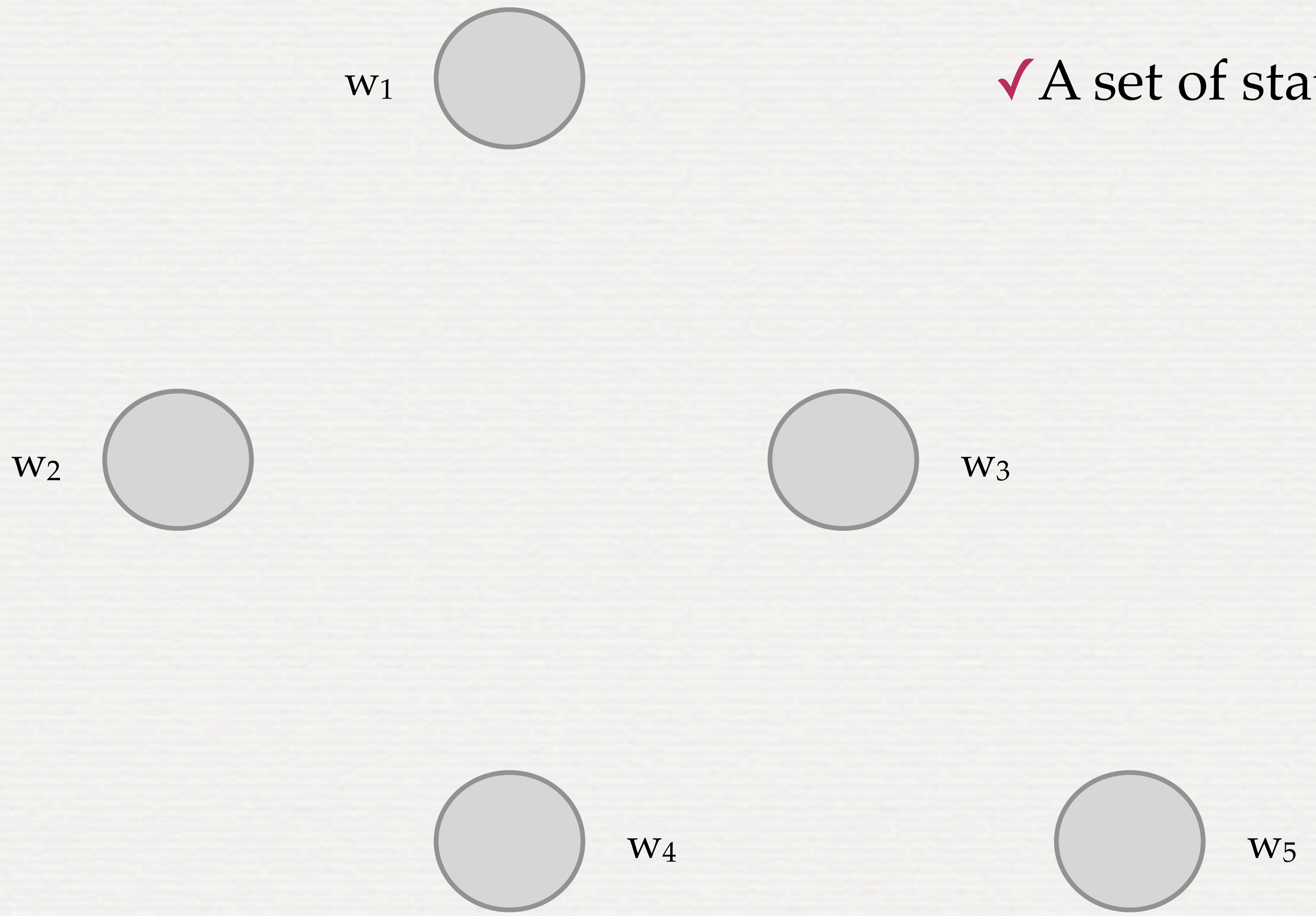
- ‘It is quite cool’ is true at this point but is **not necessarily** true (if we are out under the sun, say).
- We say that P is necessarily true if P is true in all the (relevant) situations (states, worlds, possibilities).

KRIPKE SEMANTICS

A Kripke model constitutes :

- A set of states or worlds (each one specifying truth values for all propositional variables).
- A relation on the set of states (specifying the 'relevant situations').

KRIPKE STRUCTURE



✓ A set of states

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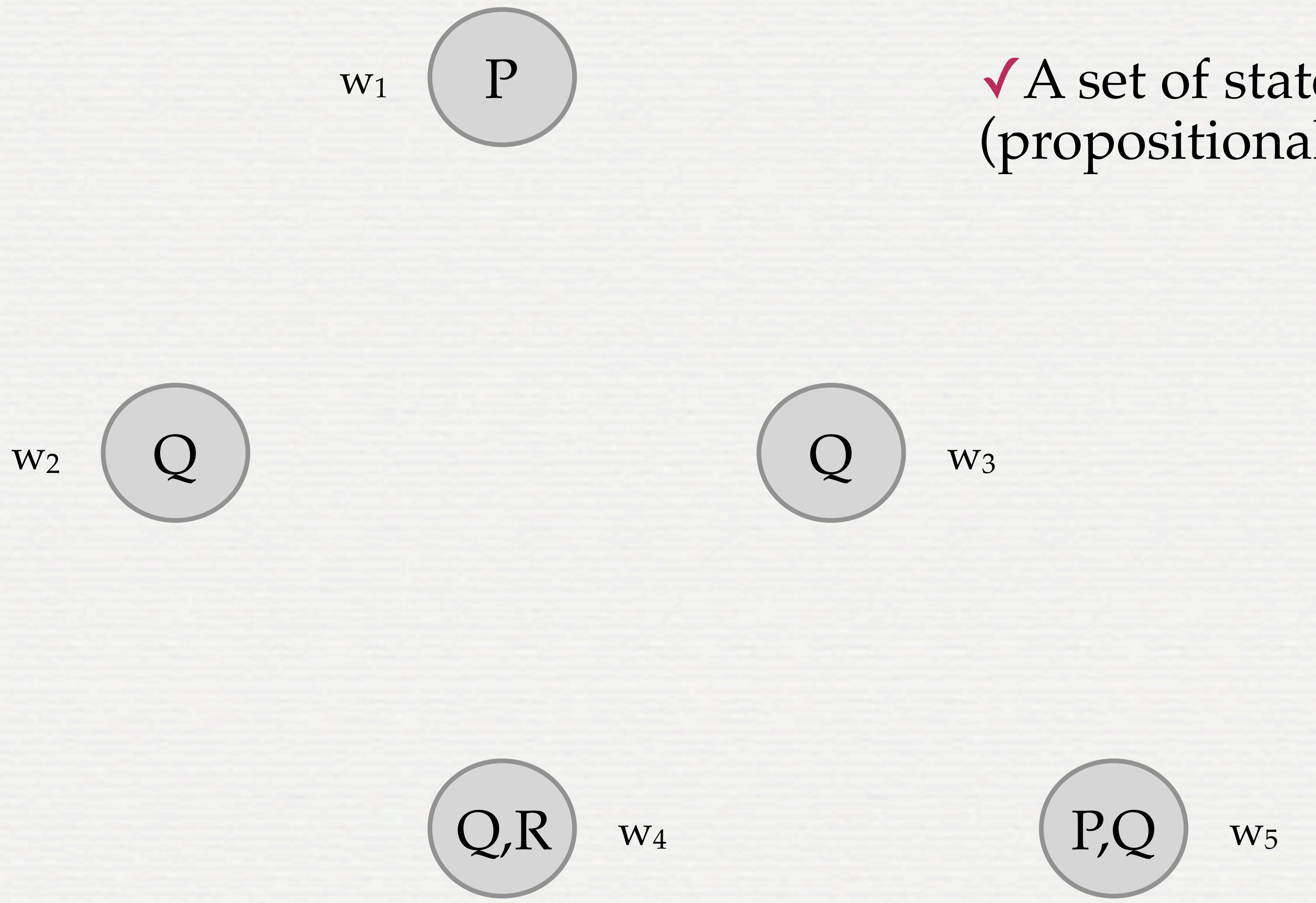

✓A set of states (propositional valuations)

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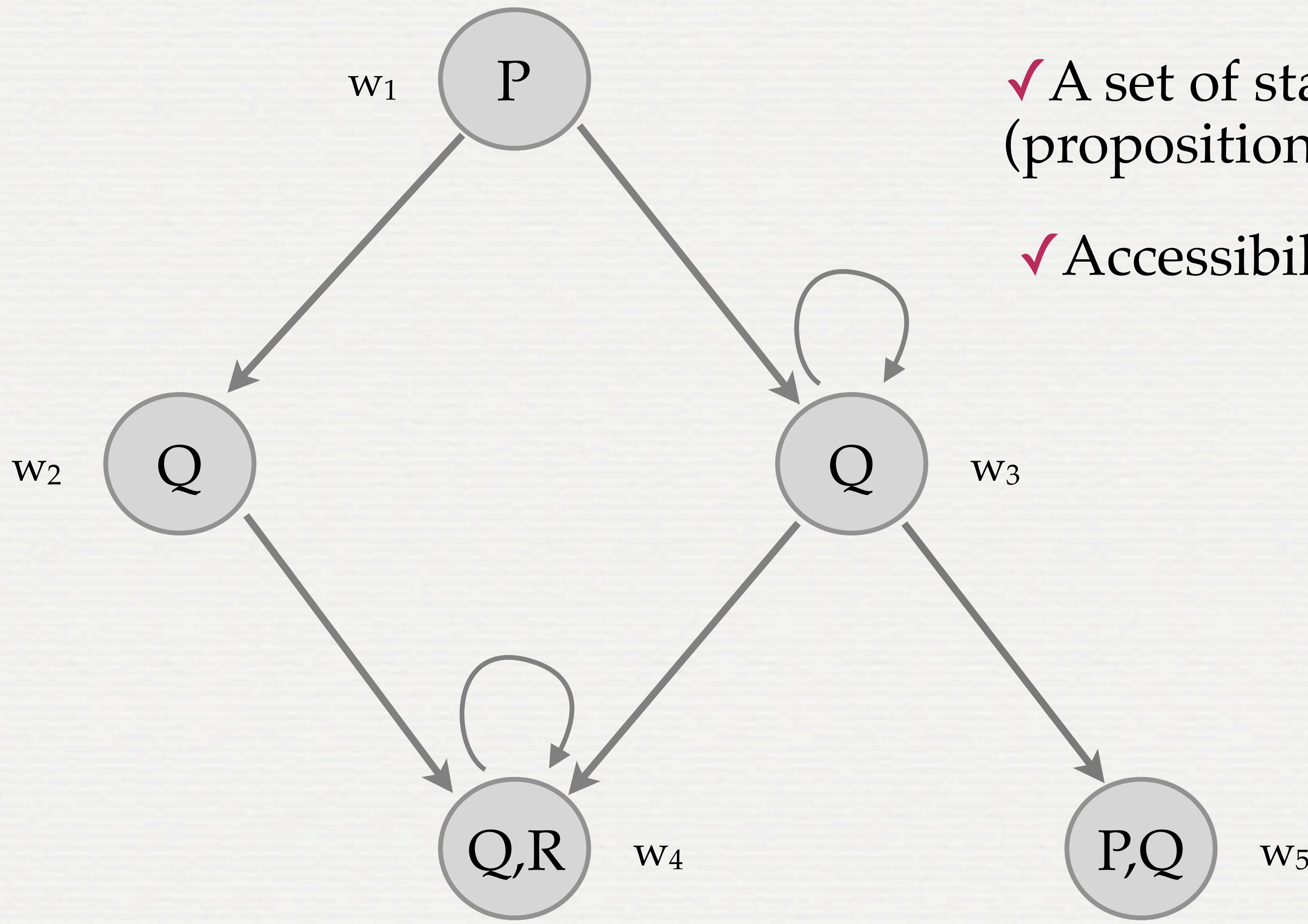

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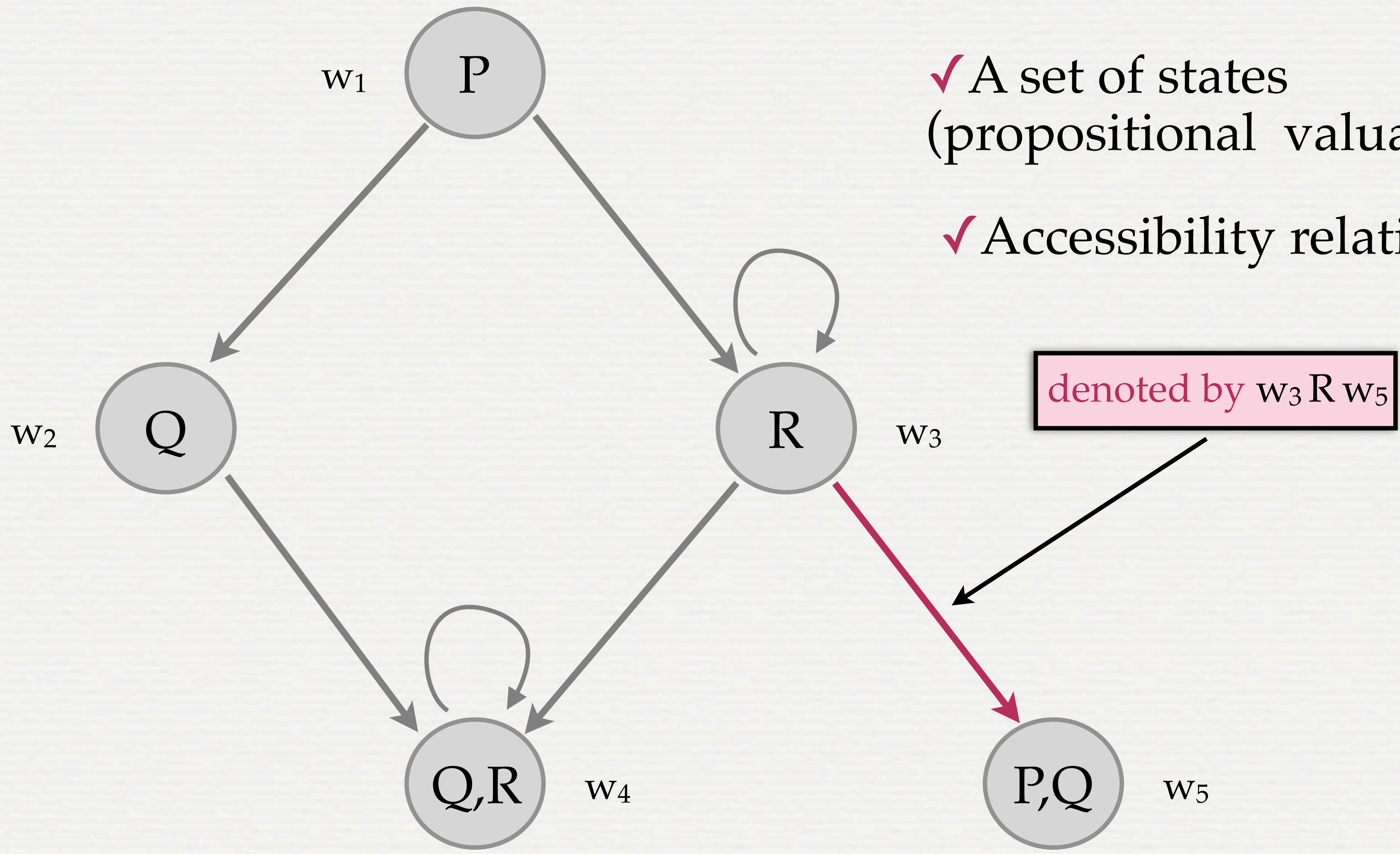

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TRUTH OF MODAL FORMULAS

Model: $\mathscr{M} = \langle W, R, V \rangle$ where $W \neq \emptyset$, $R \subseteq W \times W$ and $V\colon \mathcal{P} \rightarrow \mathbf{2}^W$ ($\mathcal{P}$ is the set of propositional variables (atomic formulas)).

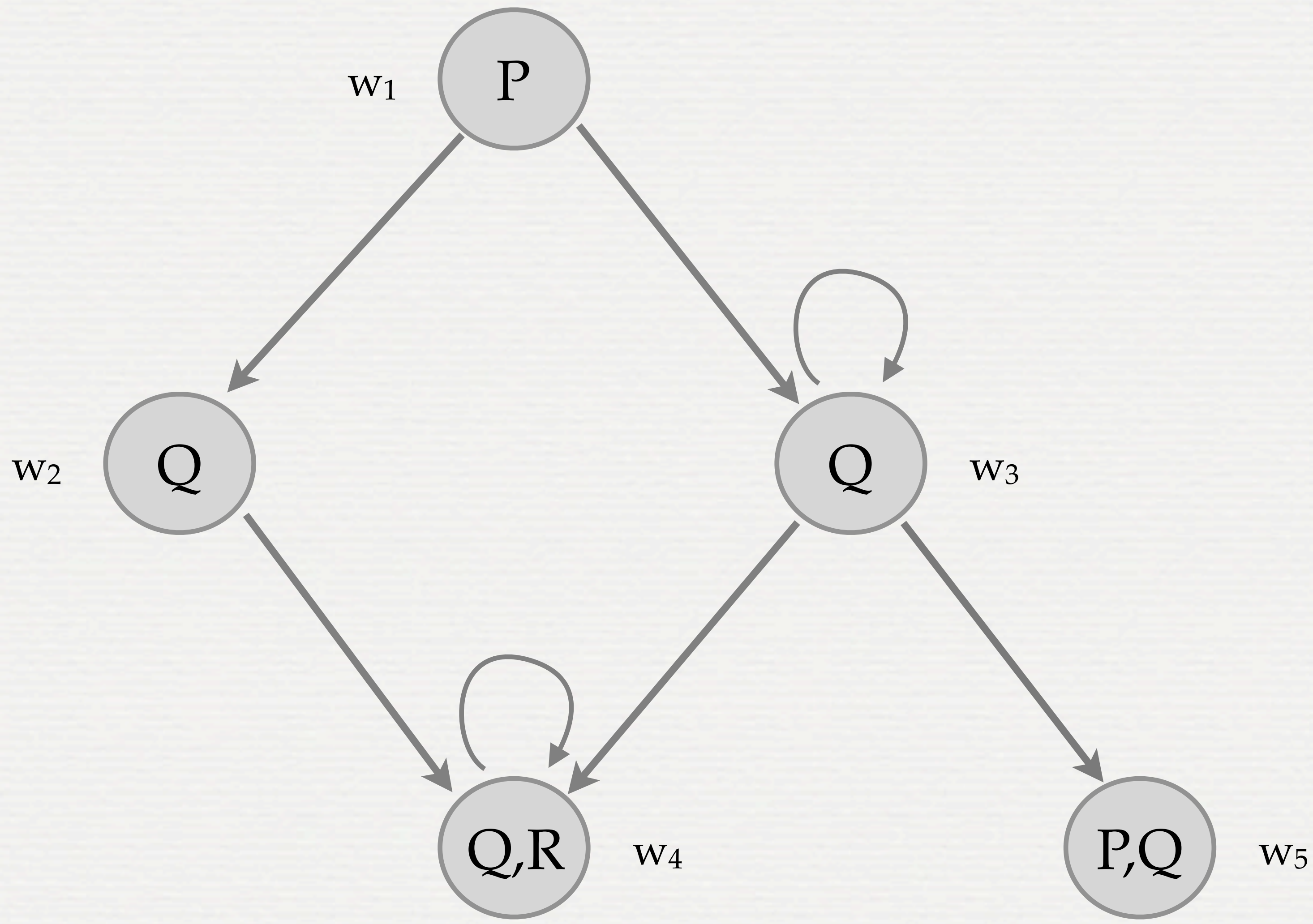
TRUTH OF MODAL FORMULAS

Model: $\mathcal{M} = \langle W, R, V\rangle$ where $W \neq \emptyset$, $R \subseteq W \times W$ and $V: \mathcal{P} \rightarrow \mathbf{2}^W$ ($\mathcal{P}$ is the set of propositional variables (atomic formulas)).

Truth of a formula at a state in a model: $\mathcal{M}, w \vDash F$

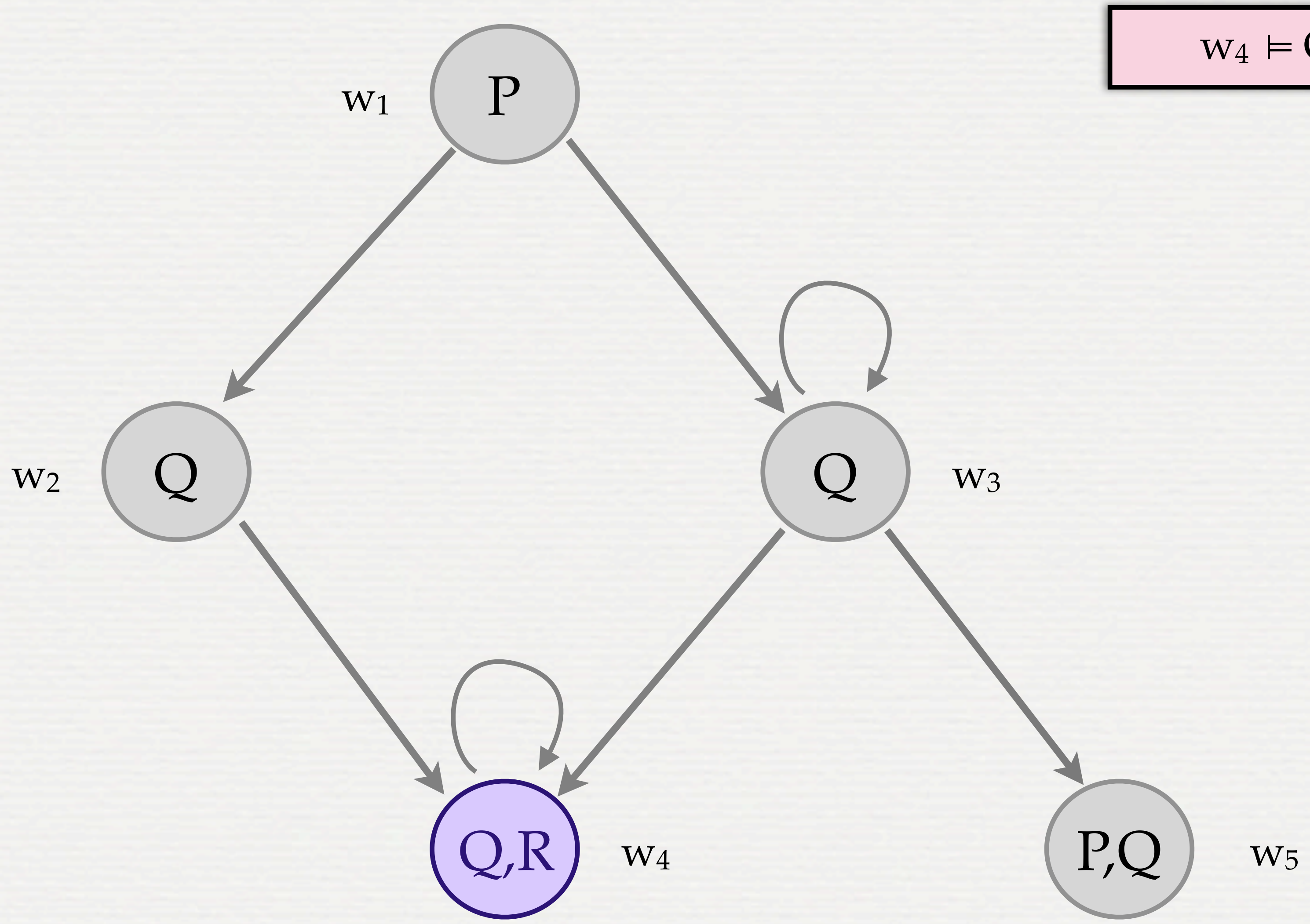
- $\mathcal{M}, w \vDash P$ iff $w \in V(P)$
- $\mathcal{M}, w \vDash \neg F$ iff $\mathcal{M}, w \nvDash F$
- $\mathcal{M}, w \vDash F \wedge G$ iff $\mathcal{M}, w \vDash F$ and $\mathcal{M}, w \vDash G$
- $\mathcal{M}, w \vDash F \vee G$ iff $\mathcal{M}, w \vDash F$ or $\mathcal{M}, w \vDash G$
- $\mathcal{M}, w \vDash \Box F$ iff for all $v \in W$ such that wRv, $\mathcal{M}, v \vDash F$
- $\mathcal{M}, w \vDash \Diamond F$ iff there exists $v \in W$ such that wRv, and $\mathcal{M}, v \vDash F$

EXAMPLE



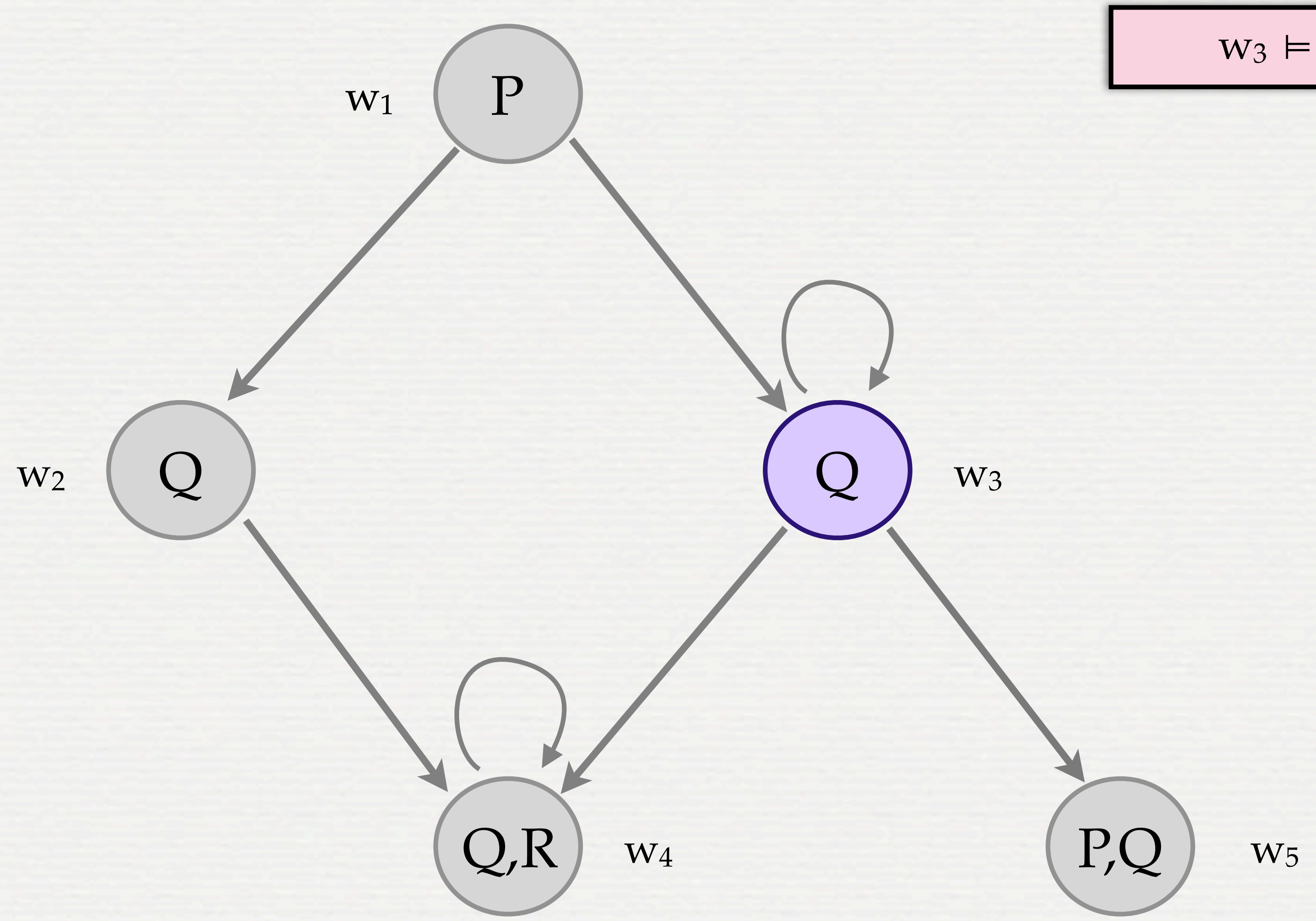
EXAMPLE

$w_4 \vDash Q \wedge R$



EXAMPLE

$w_3 \vDash \Box Q$



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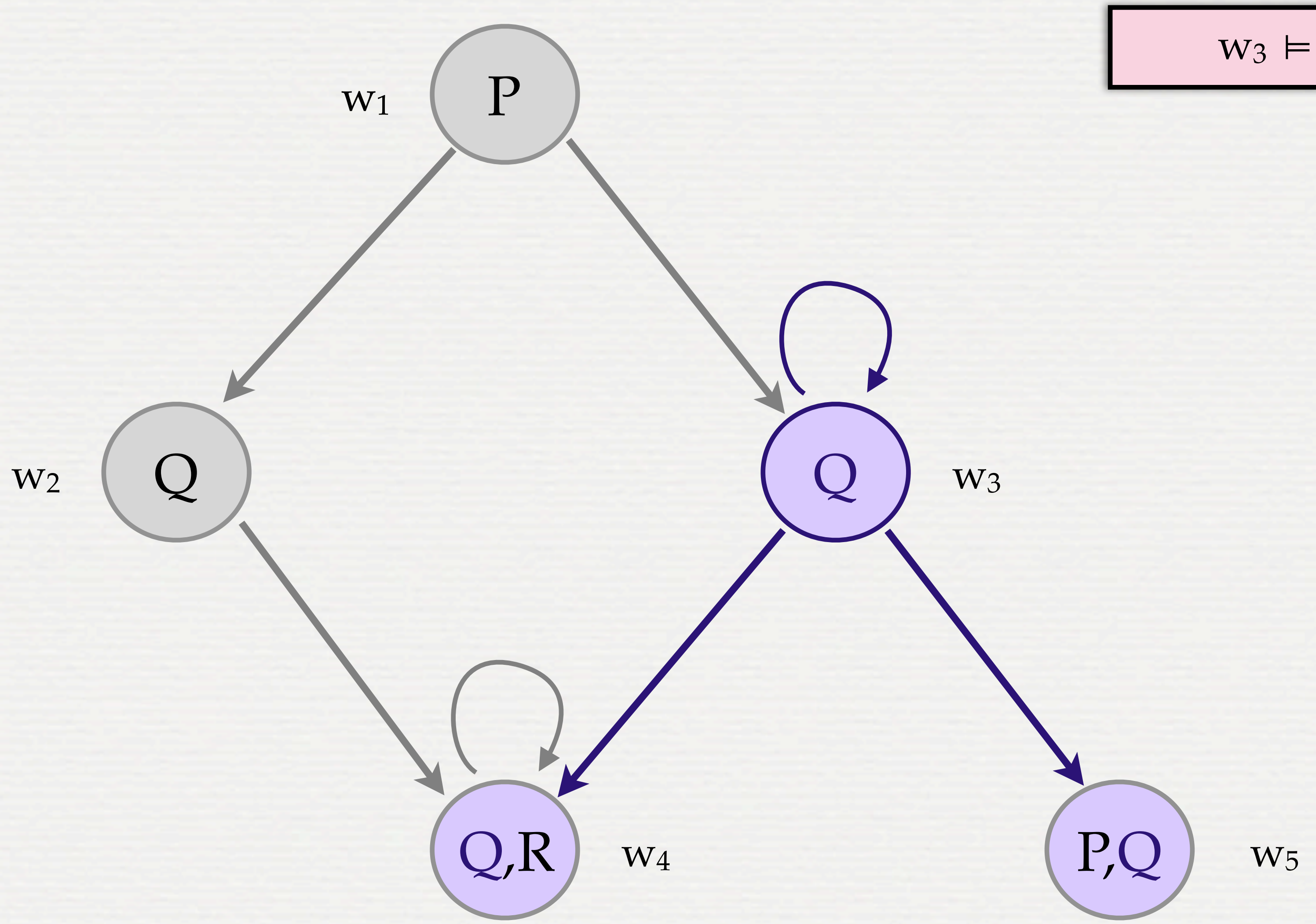

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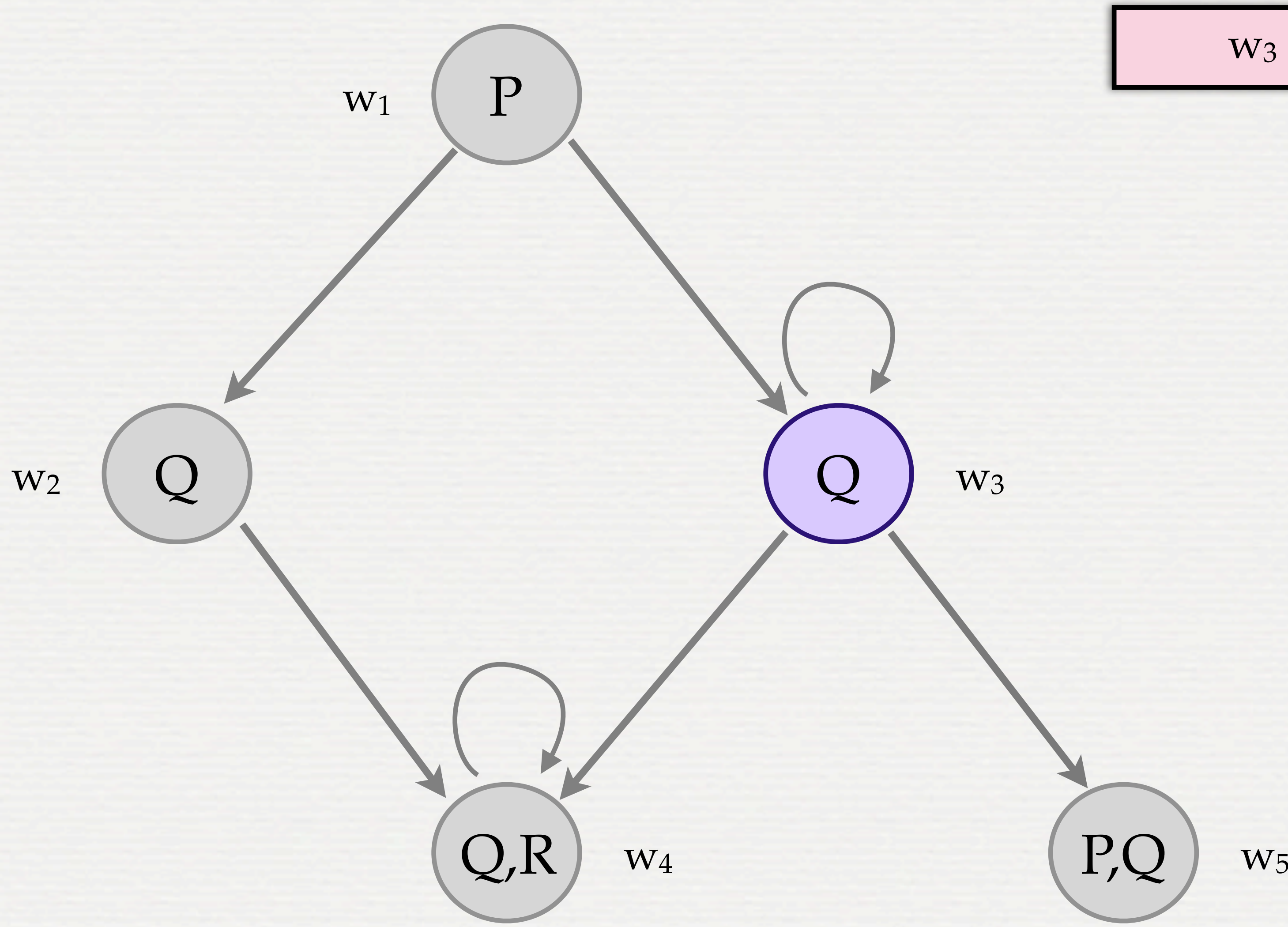

$w_3 \models \Diamond R$

EXAMPLE

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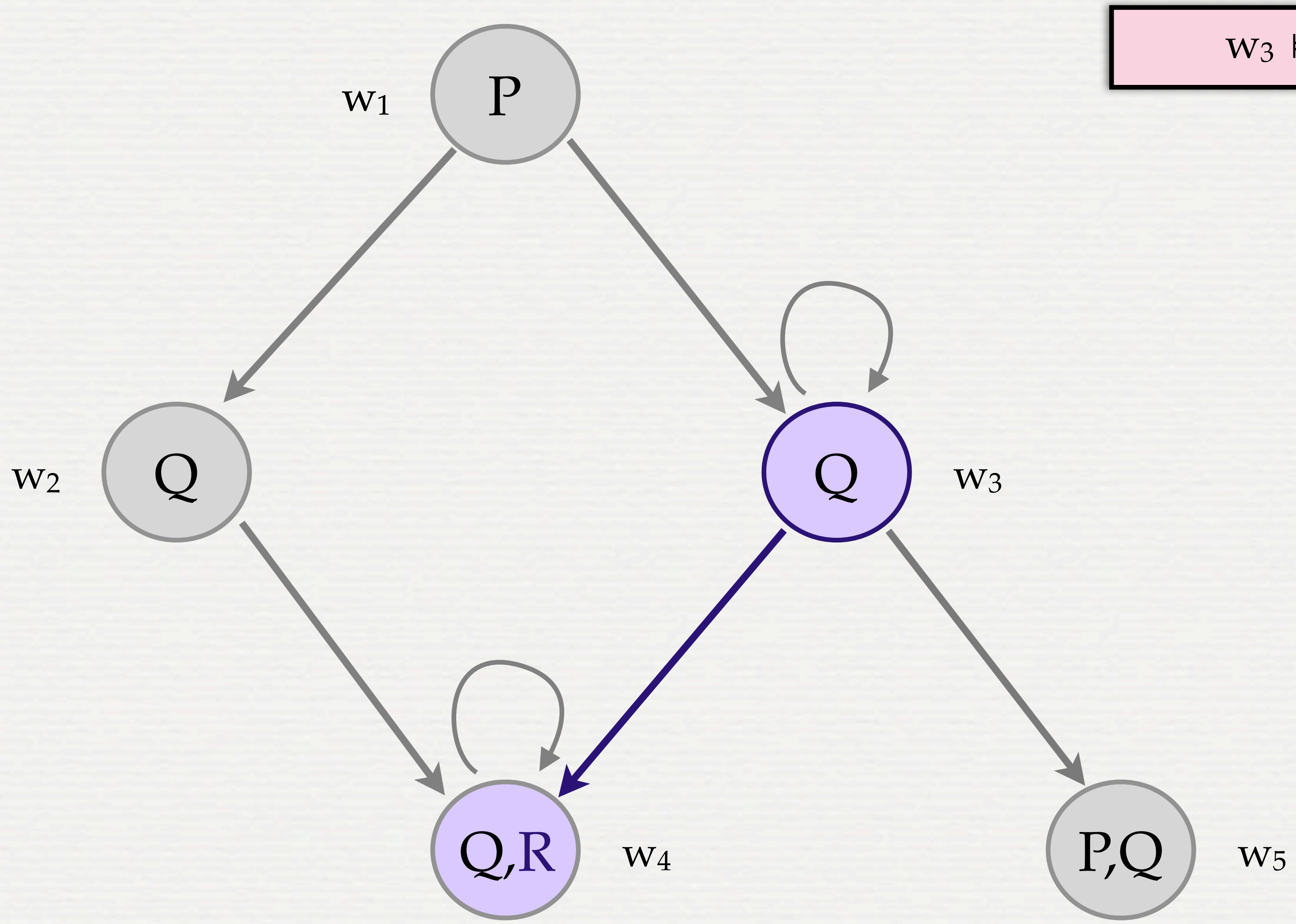
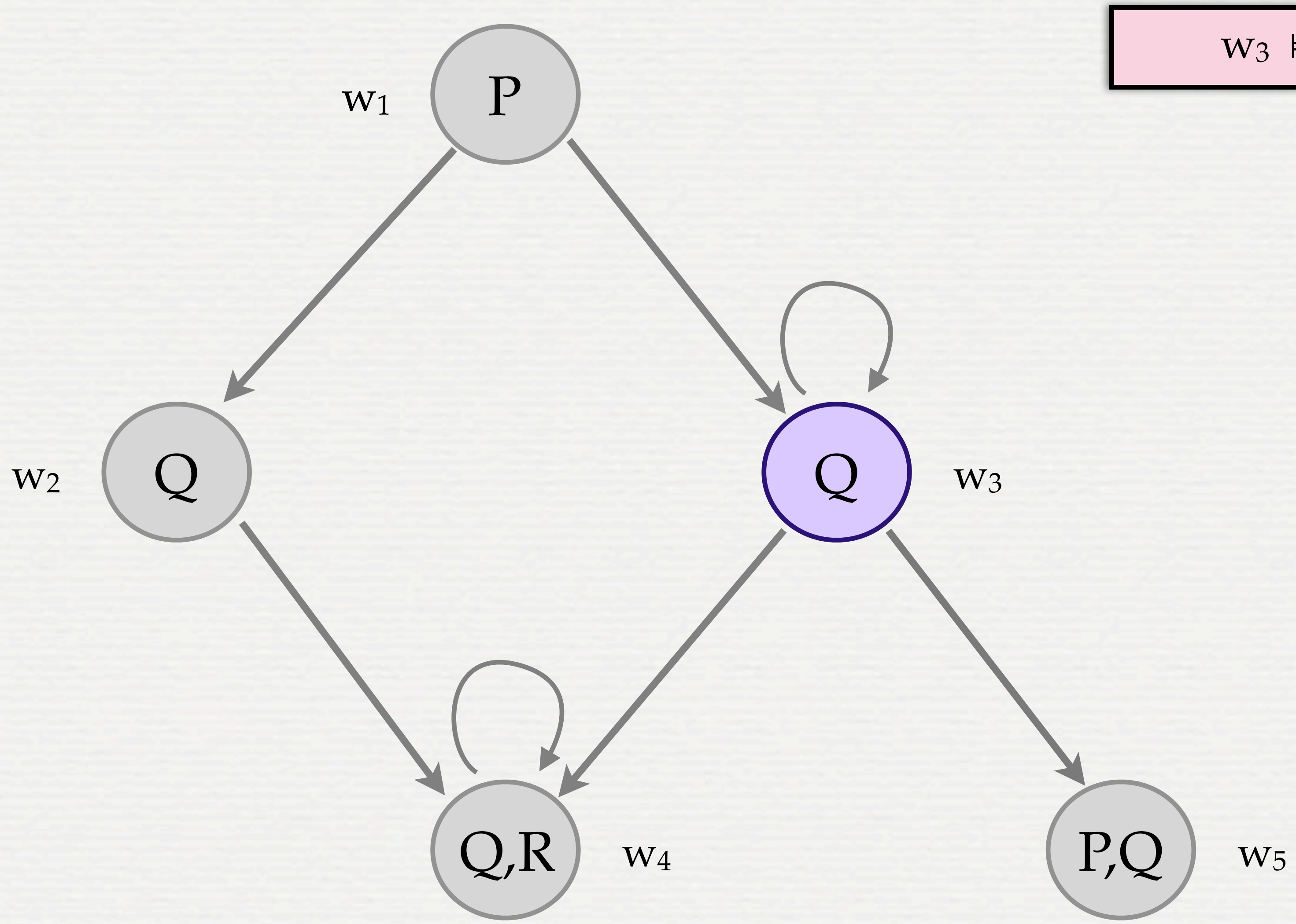


EXAMPLE

$w_3 \not\models \Box R$



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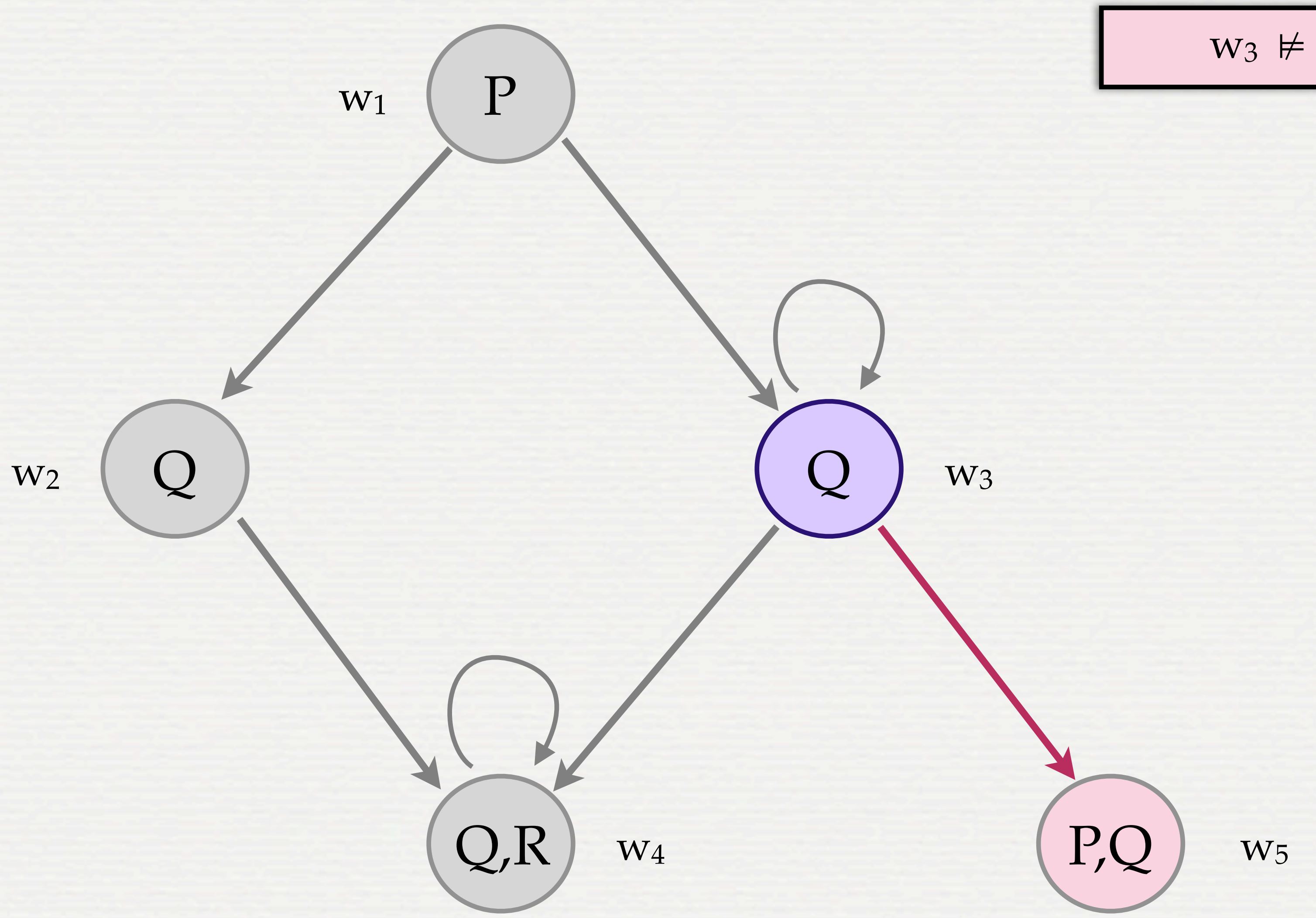

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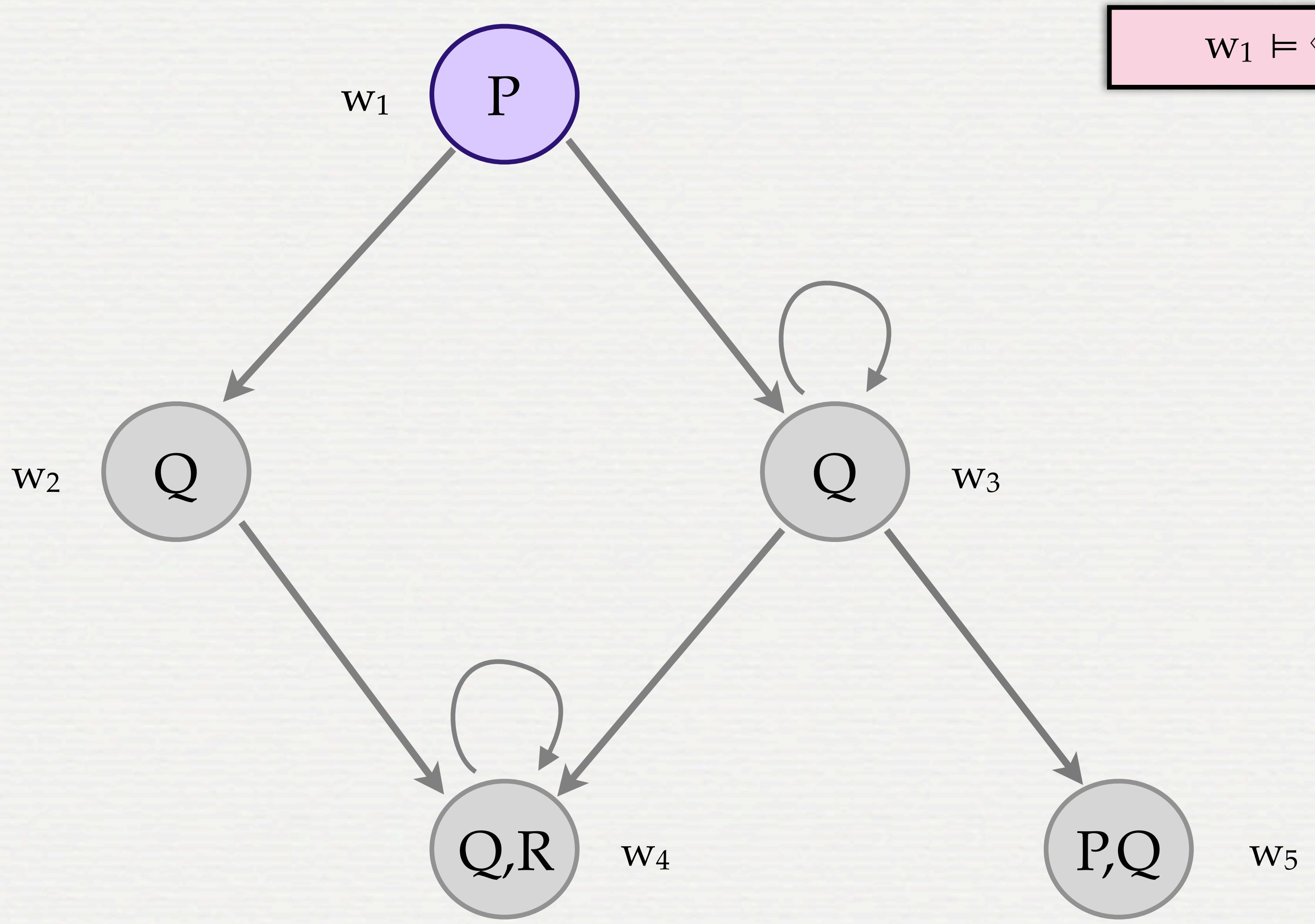

$w_1 \models \Diamond\Box Q$

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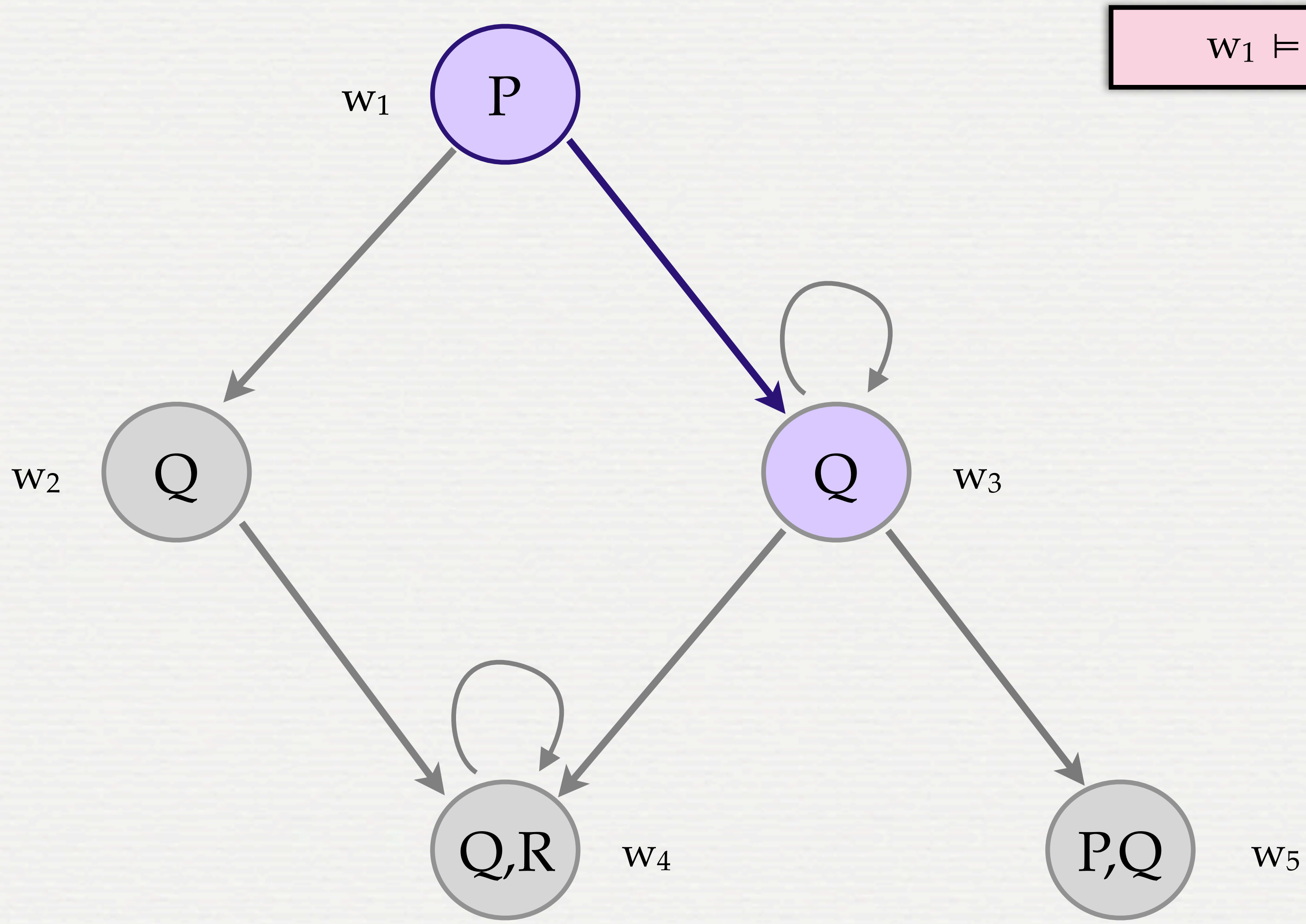

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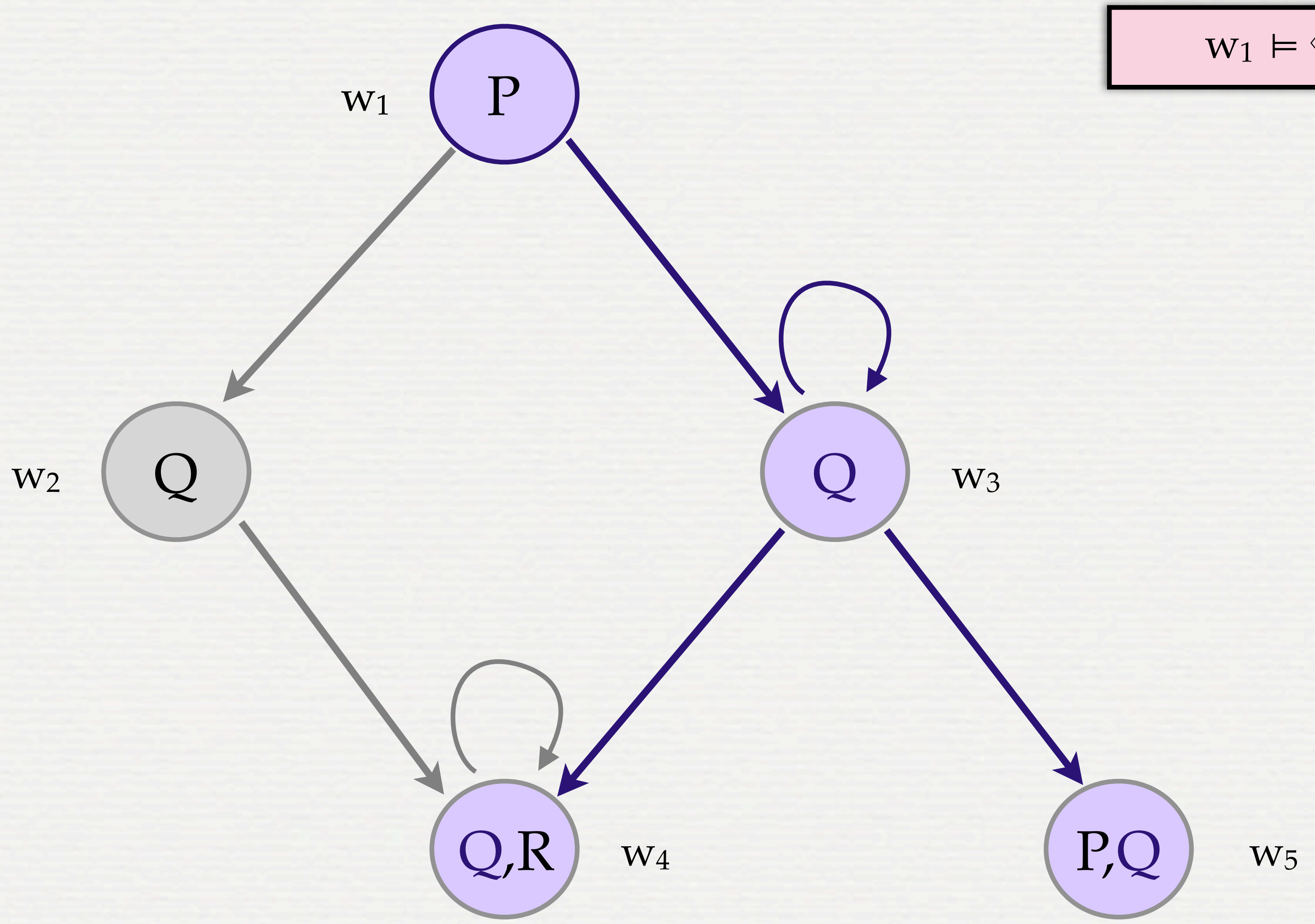

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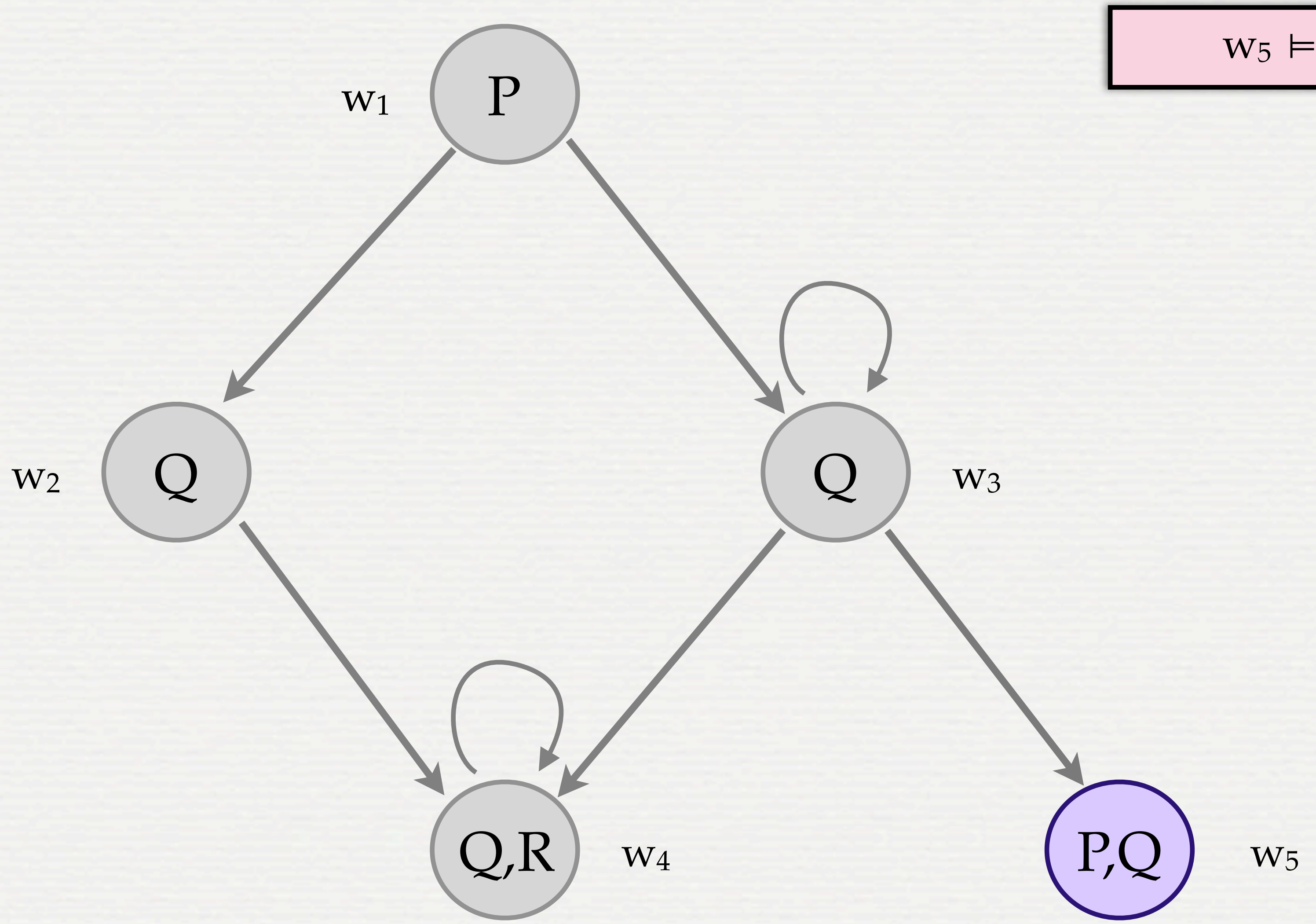
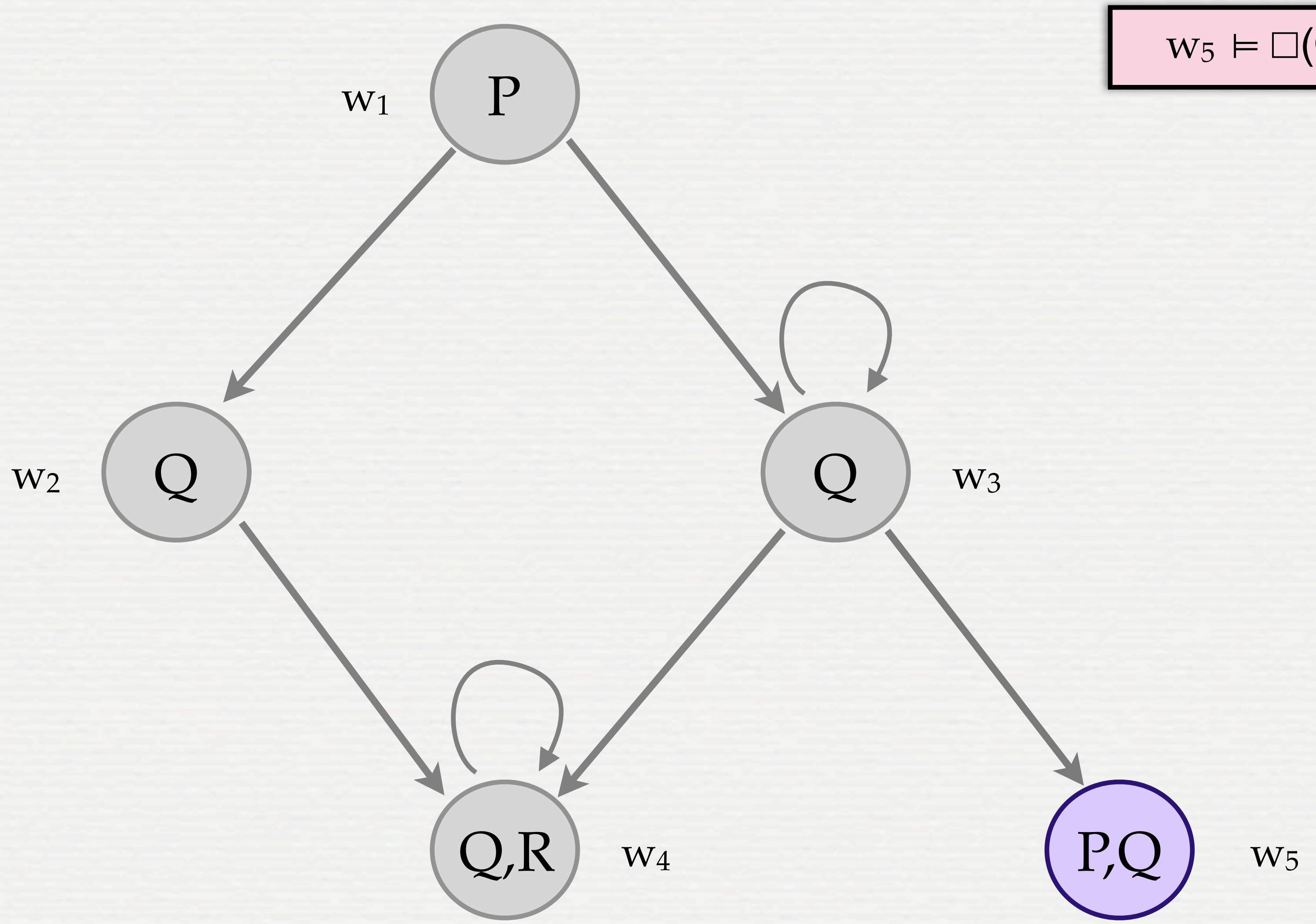


$w_5 \models \Box R$

EXAMPLE

$w_5 \models \Box(Q \land \neg Q)$



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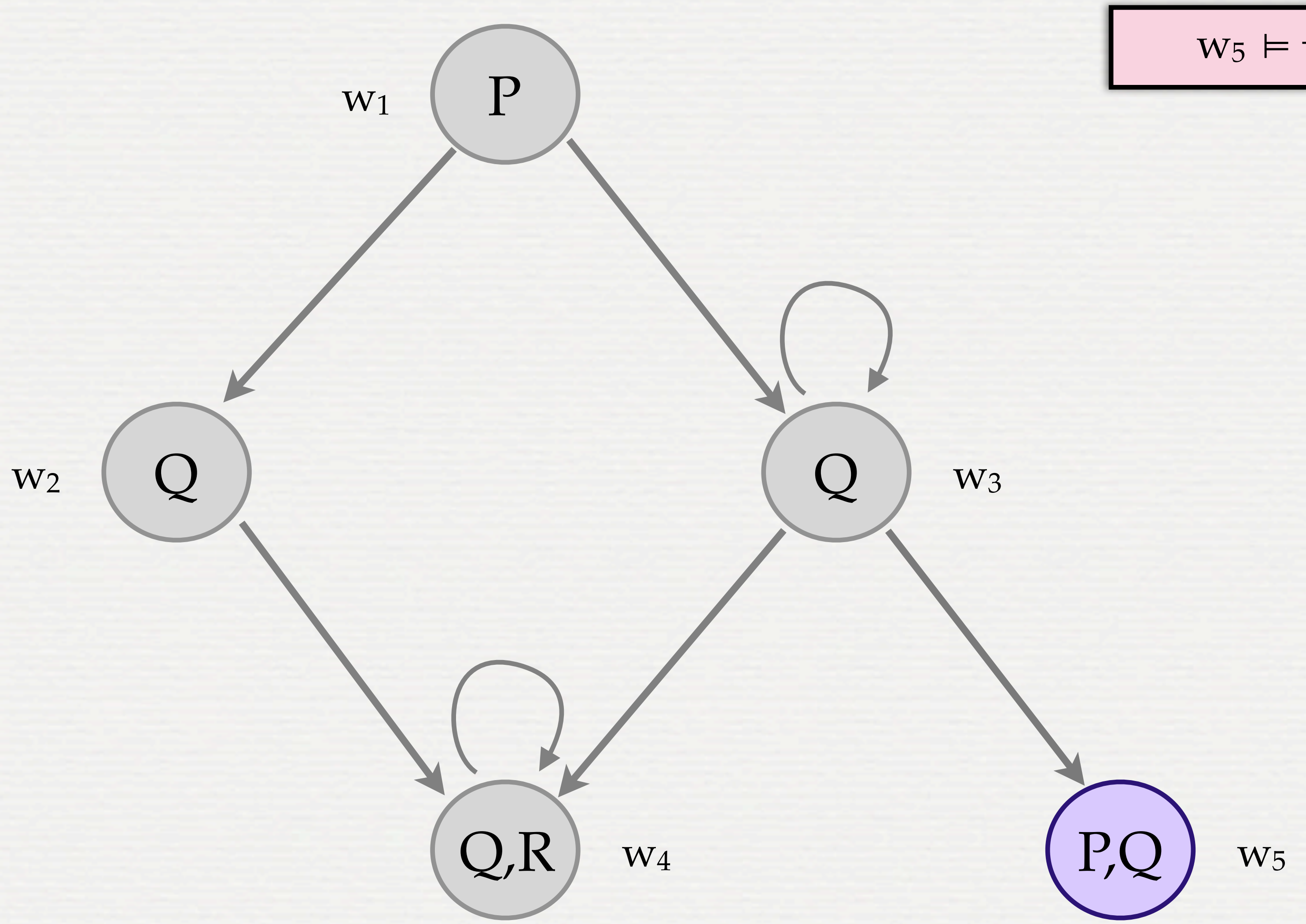

$w_5 \vDash \neg\Diamond\neg Q$

SOME FACTS/QUESTIONS

- $\Box F \lor \neg\Box F$ is always true (that is, true at any state of any Kripke model), but what about $\Box F \lor \Box \neg F$?

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- $\Box F \vee \neg \Box F$ is always true (that is, true at any state of any Kripke model), but what about $\Box F \vee \Box \neg F$?
- Is $\Box(F \wedge G) \rightarrow (\Box F \wedge \Box G)$ always true? What about $\Box(F \vee G) \rightarrow (\Box F \vee \Box G)$?

SOME FACTS/QUESTIONS

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- Is $\Box(F \land G) \to (\Box F \land \Box G)$ always true? What about $\Box(F \lor G) \to (\Box F \lor \Box G)$?
- Is $\Box F \leftrightarrow \neg\Diamond\neg F$ always true?

SATISFACTION AND VALIDITY

- A modal formula F is said to be satisfiable if there exists a model $\mathscr{M} = \langle W, R, V \rangle$ and a state $w \in W$ such that $\mathscr{M}, w \models F$.

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- A modal formula F is said to be true in a model $\mathcal{M} = \langle W, R, V\rangle$, denoted by $\mathcal{M} \models F$, if $\mathcal{M}, w \models F$ for all states $w \in W$.
- A modal formula F is said to be valid, denoted by $\models F$, if $\mathcal{M} \models F$ for all models $\mathcal{M}$.

LET'S WORK ON BOARD !

H.W.

- $\Box F \leftrightarrow \neg\Diamond\neg F$ is a valid formula.
- $\Box(F \wedge G) \rightarrow (\Box F \wedge \Box G)$ is a valid formula.
- $\Box(F \vee G) \rightarrow (\Box F \vee \Box G)$ is not a valid formula.

Another perspective: Checking truth of modal formulas in these models can be thought of as a dynamic procedure.

MODAL EVALUATION GAME

Model: $\mathscr{M} = \langle W, R, V \rangle$, Formula: F

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Game: G ($\mathscr{M}$, w, F)

Players: Verifier (E), Falsifier (A)

Verifier (E) ↓ F is true

Falsifier (A) ↓ F is false

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- ◇F: E chooses an R successor v of the current world and the play continues with F at v

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Game: G ($\mathscr{M}$, w, F)

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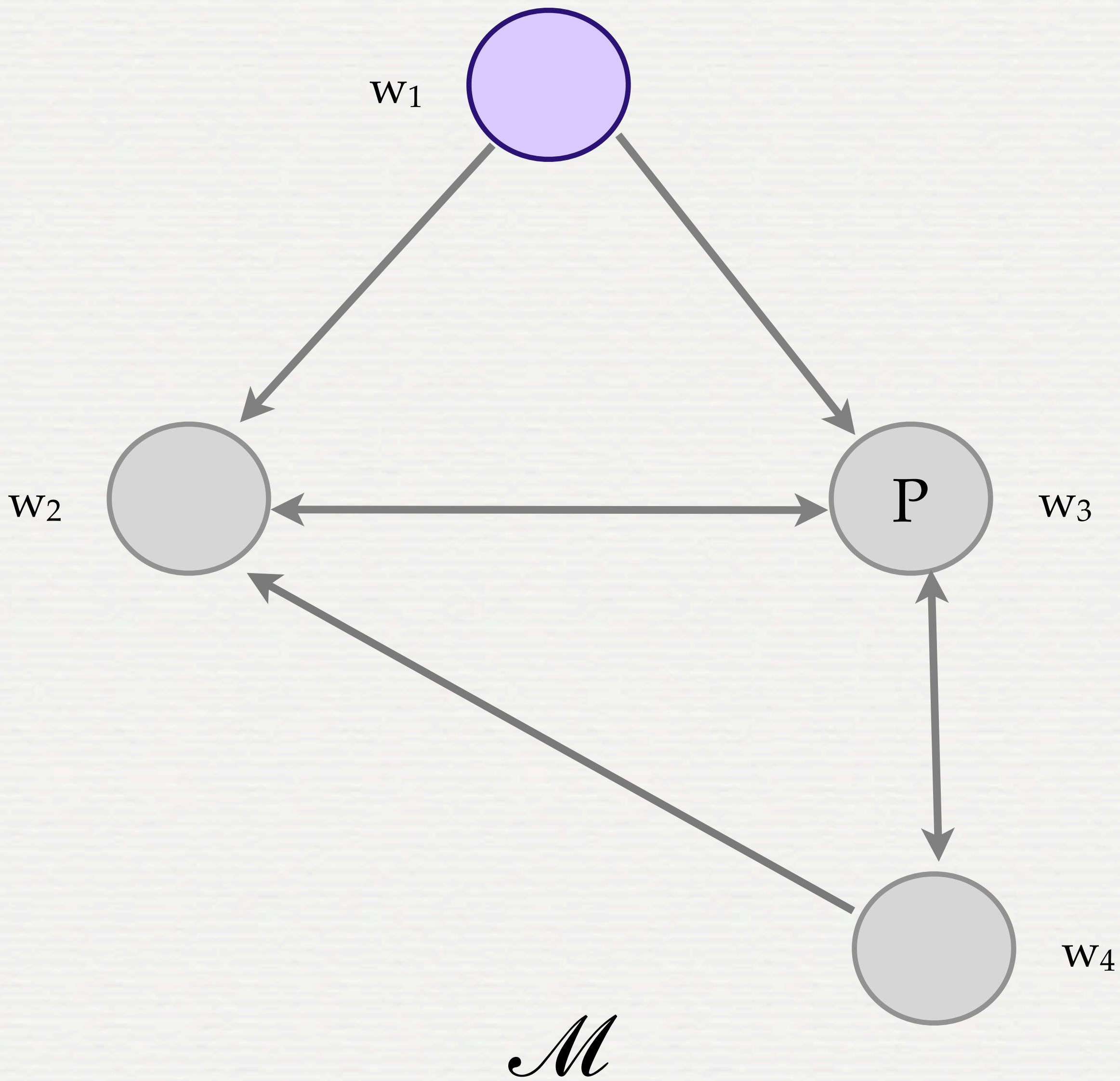
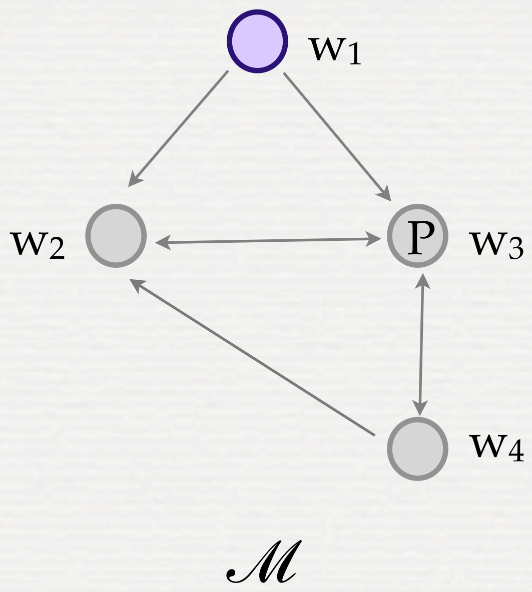
- □F: A chooses an R successor v of the current world and the play continues with F at v
- ◇F: E chooses an R successor v of the current world and the play continues with F at v
- ¬F: players switch roles in the F game

EXAMPLE



F: $\Box(\Diamond P \vee \Box\Diamond P)$

Players: E, A



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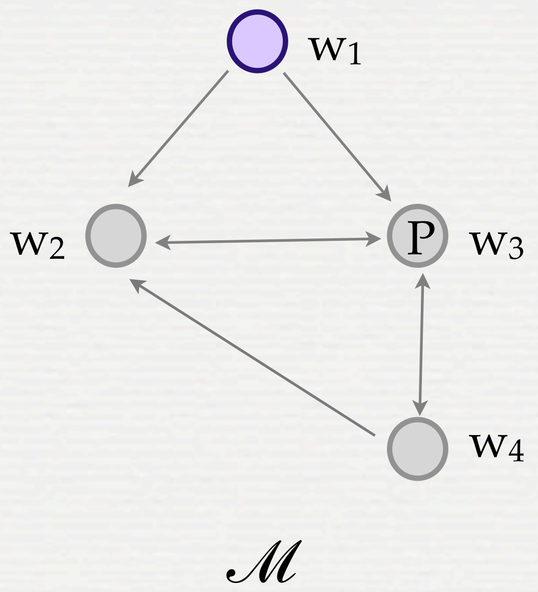
w_1, $\Box(\Diamond P \vee \Box\Diamond P)$



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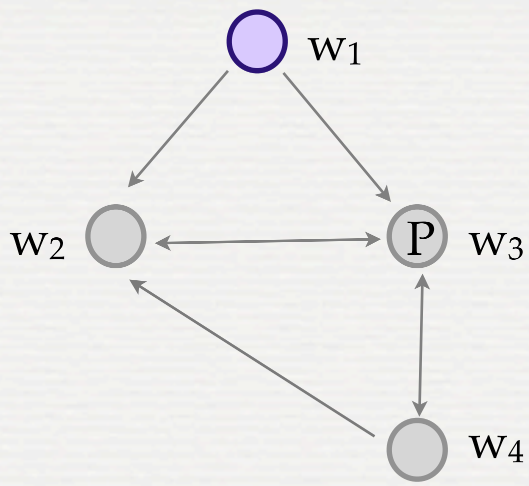


$\mathscr{M}$

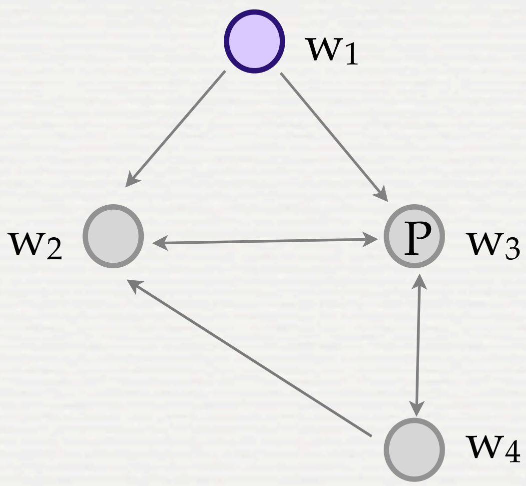
F: □(◇P ∨ □◇P)

Players: E, A

w_1, □(◇P ∨ □◇P)

w_2, ◇P ∨ □◇P

w_3, ◇P ∨ □◇P



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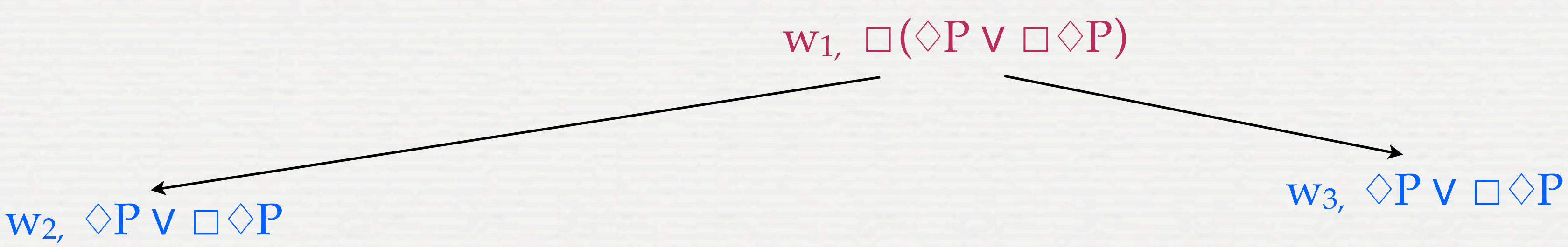
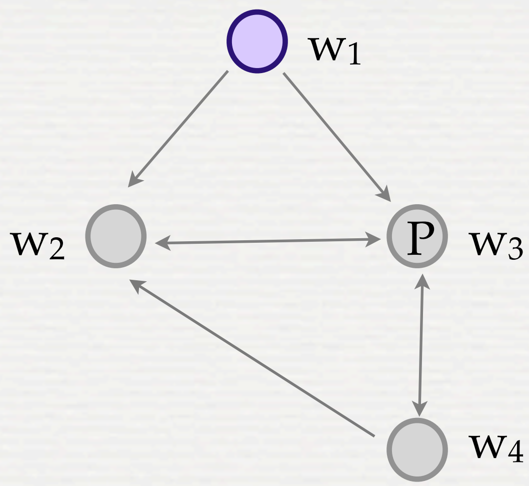
Players: E, A





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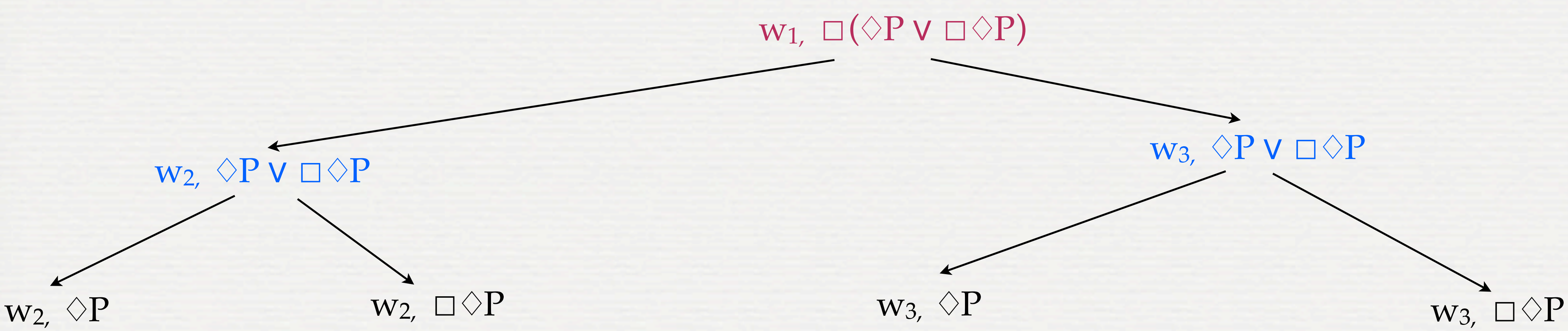
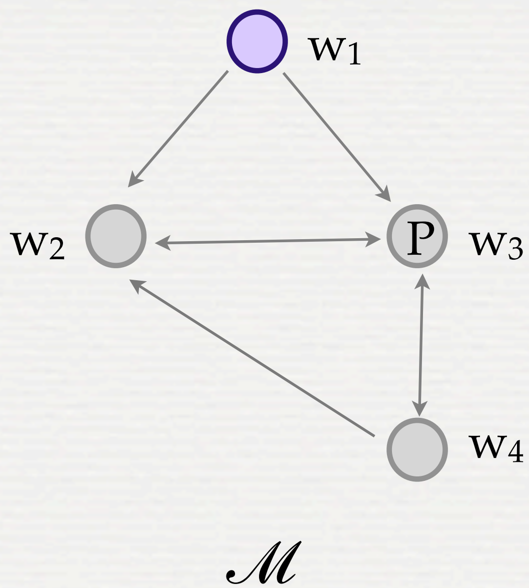
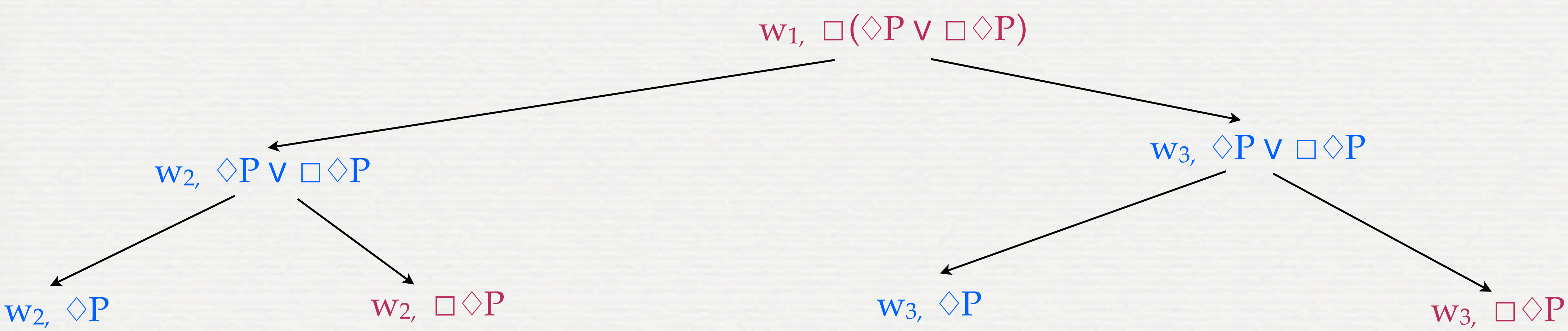
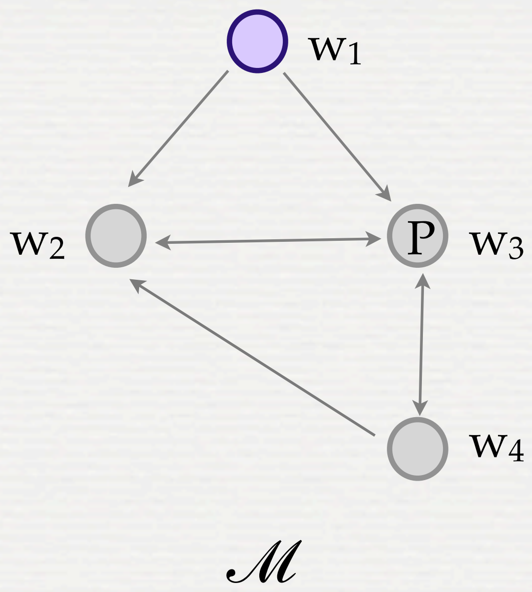




F: $\Box(\Diamond P \lor \Box\Diamond P)$

Players: E, A





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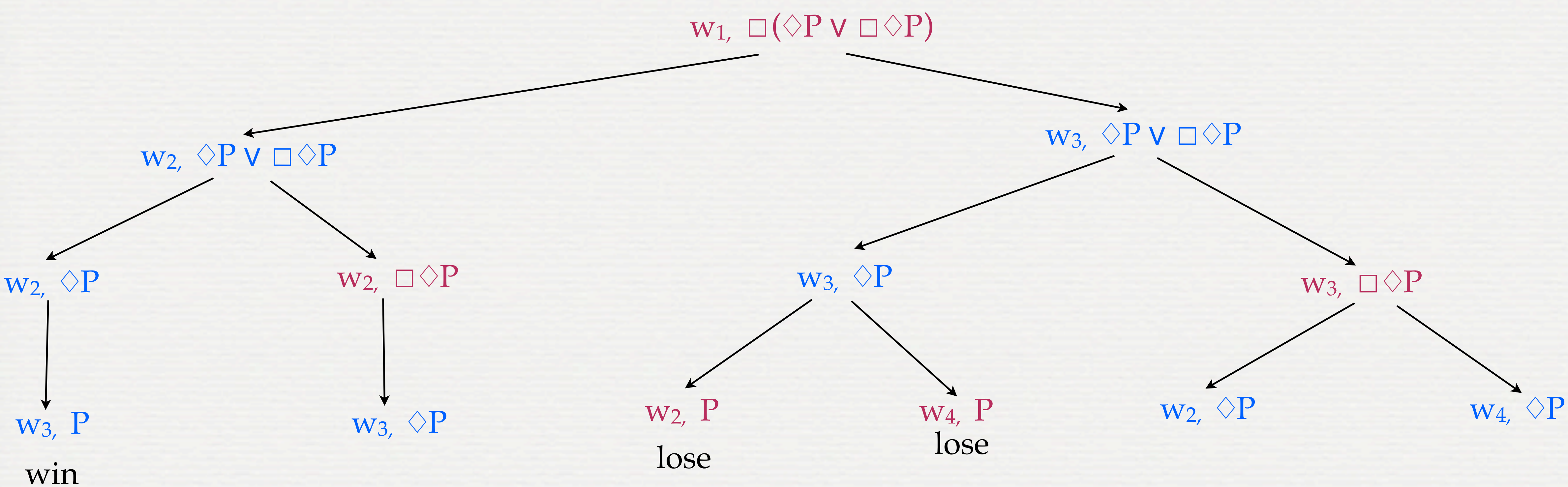

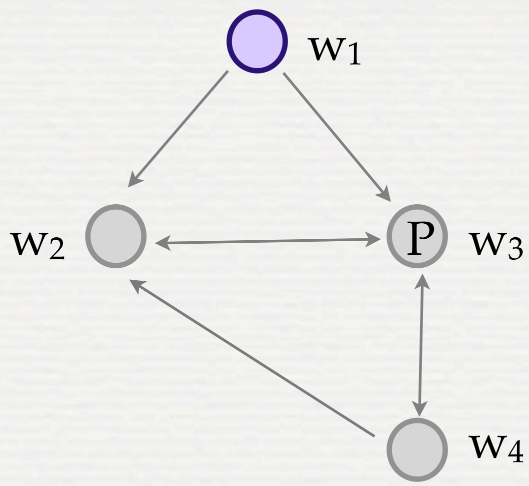

$\mathscr{M}$

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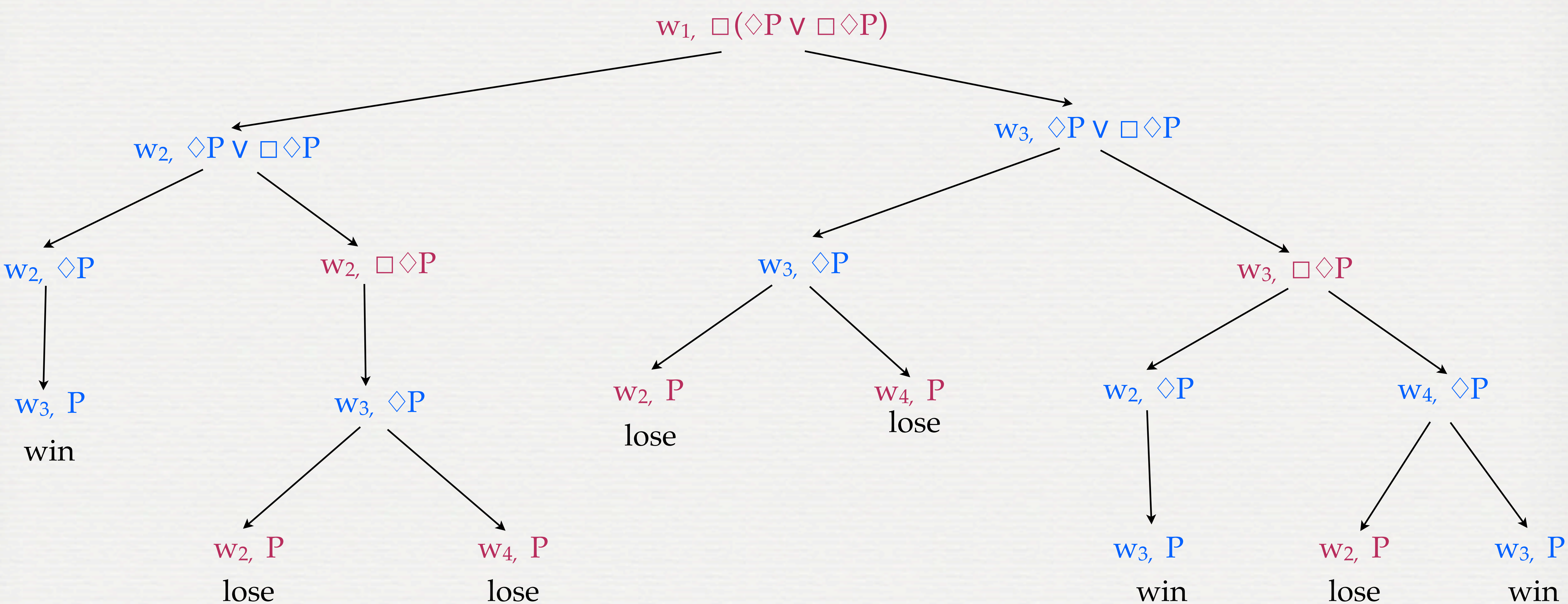

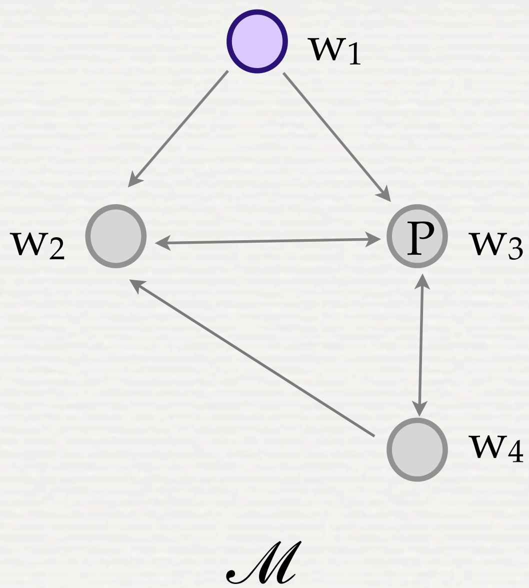

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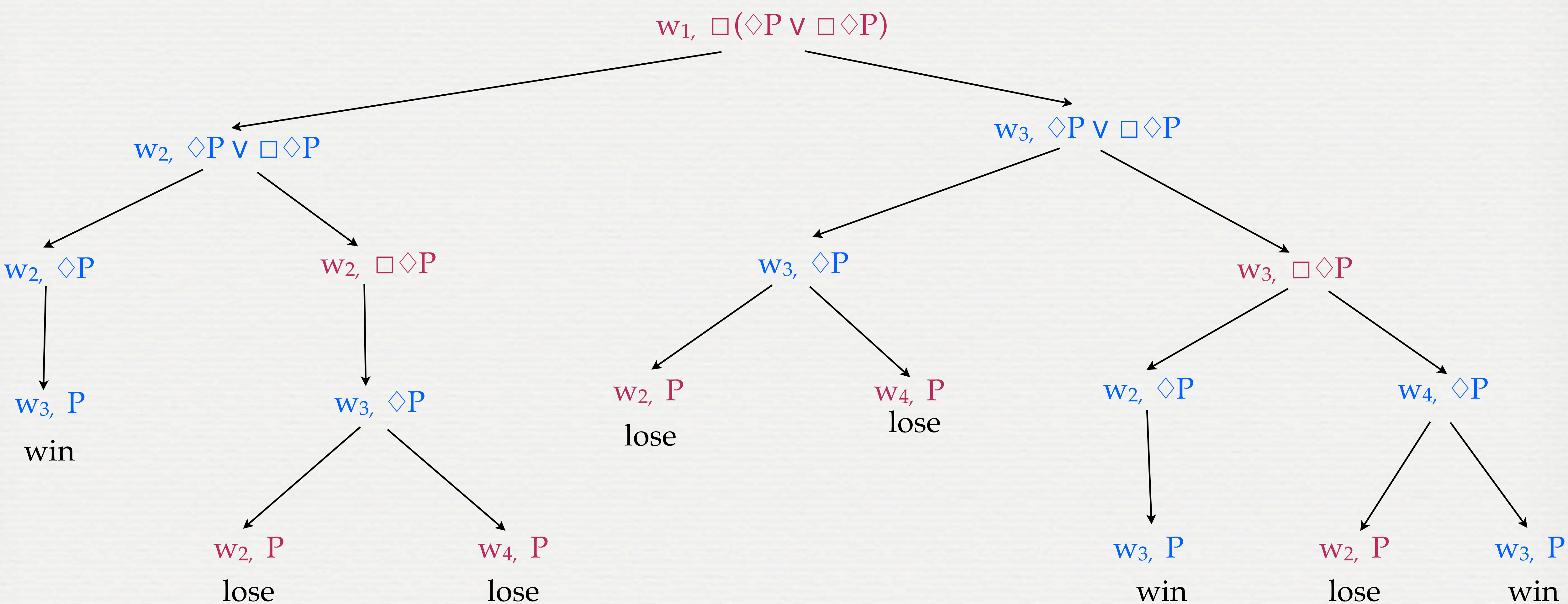
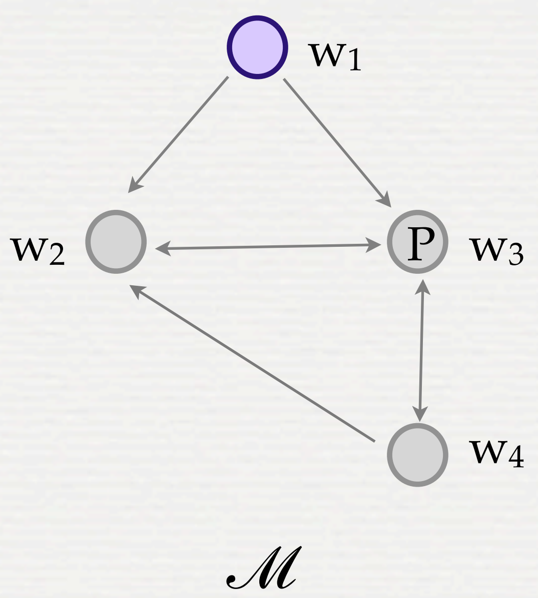


Does E have a winning strategy?



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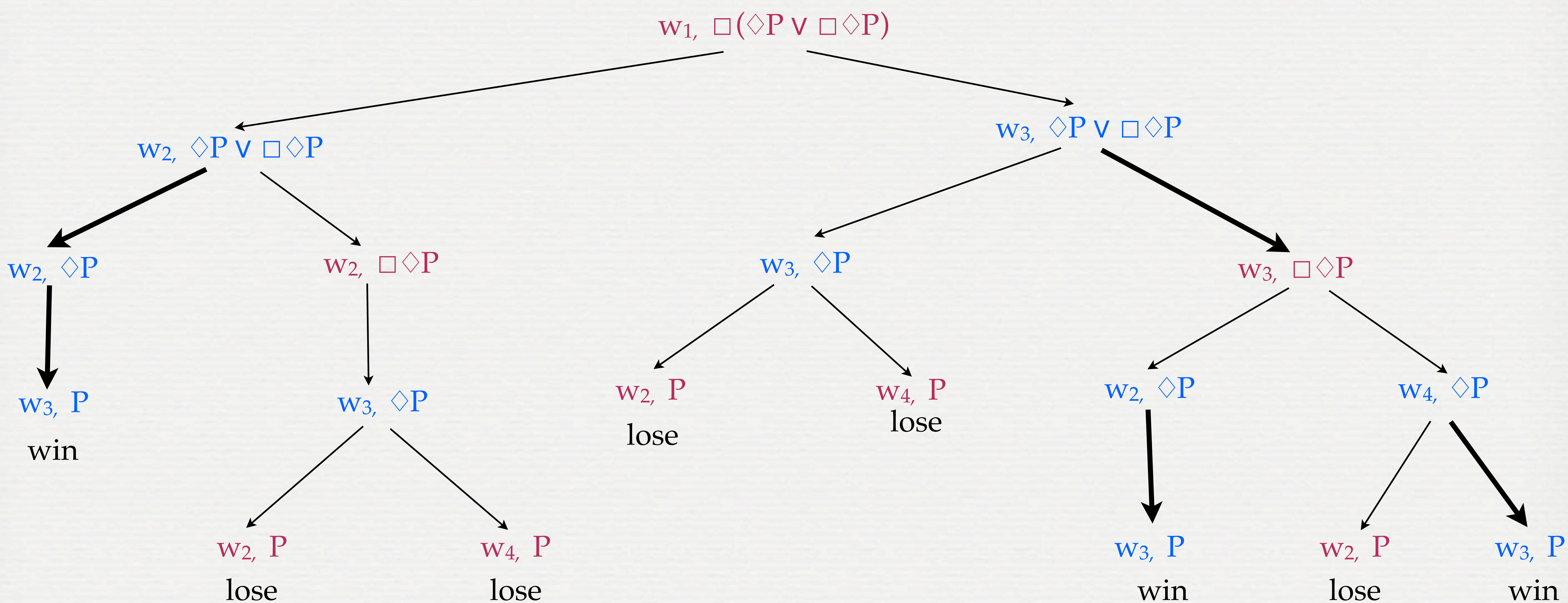


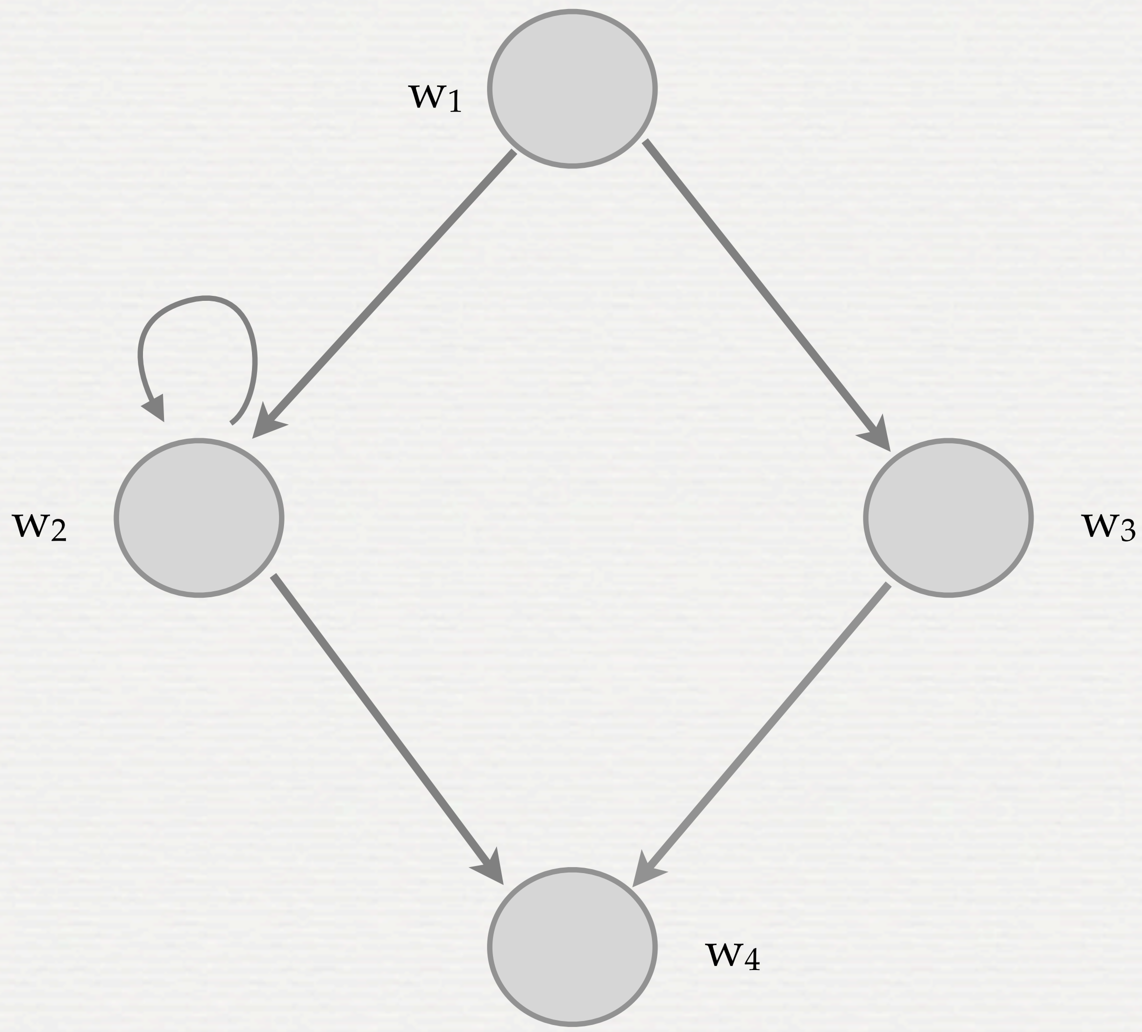
E does !

TFAE

- F is true in $\mathscr{M}$ at w.
- E has a winning strategy in the game G ($\mathscr{M}$, w, F).

DEFINING STATES

DEFINING STATES



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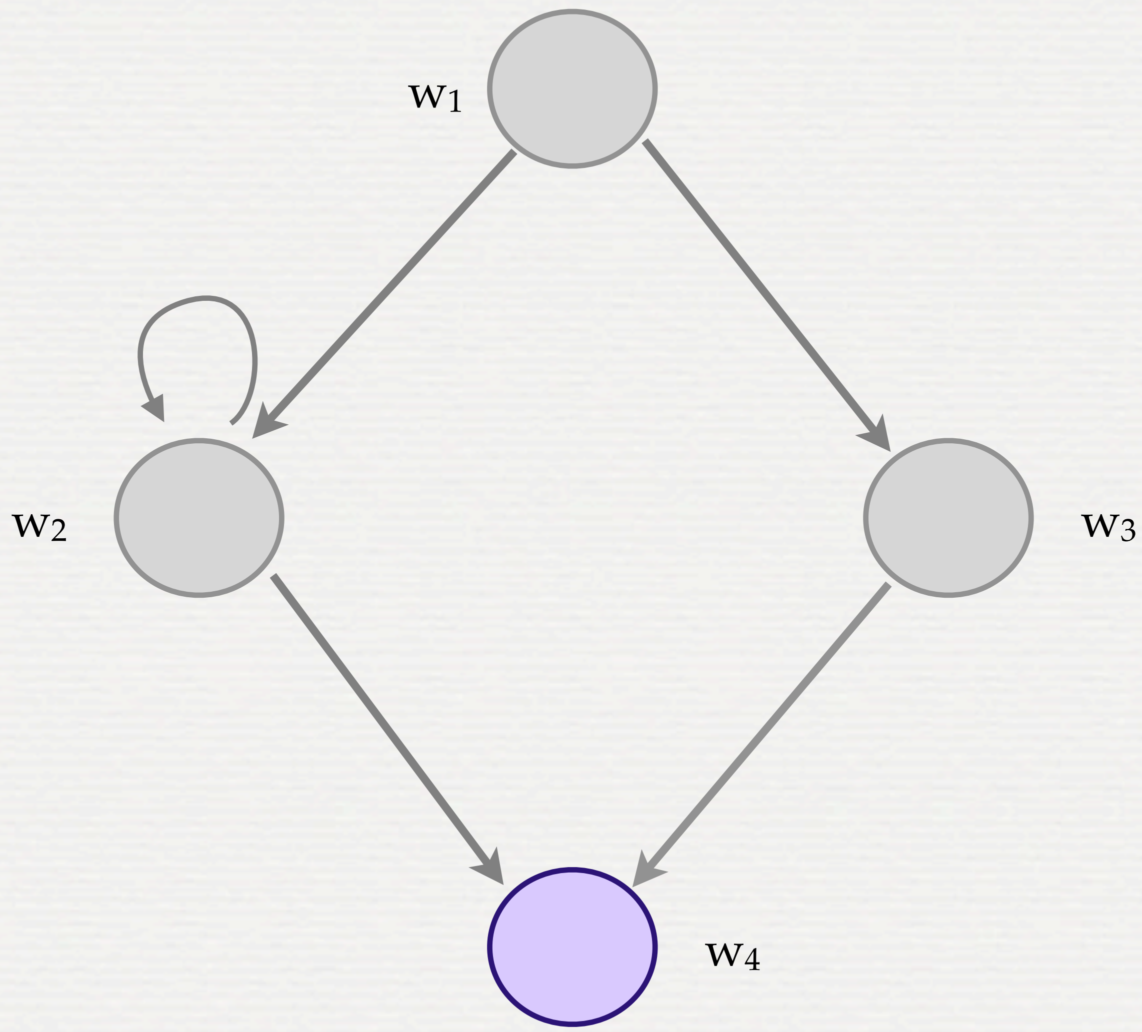
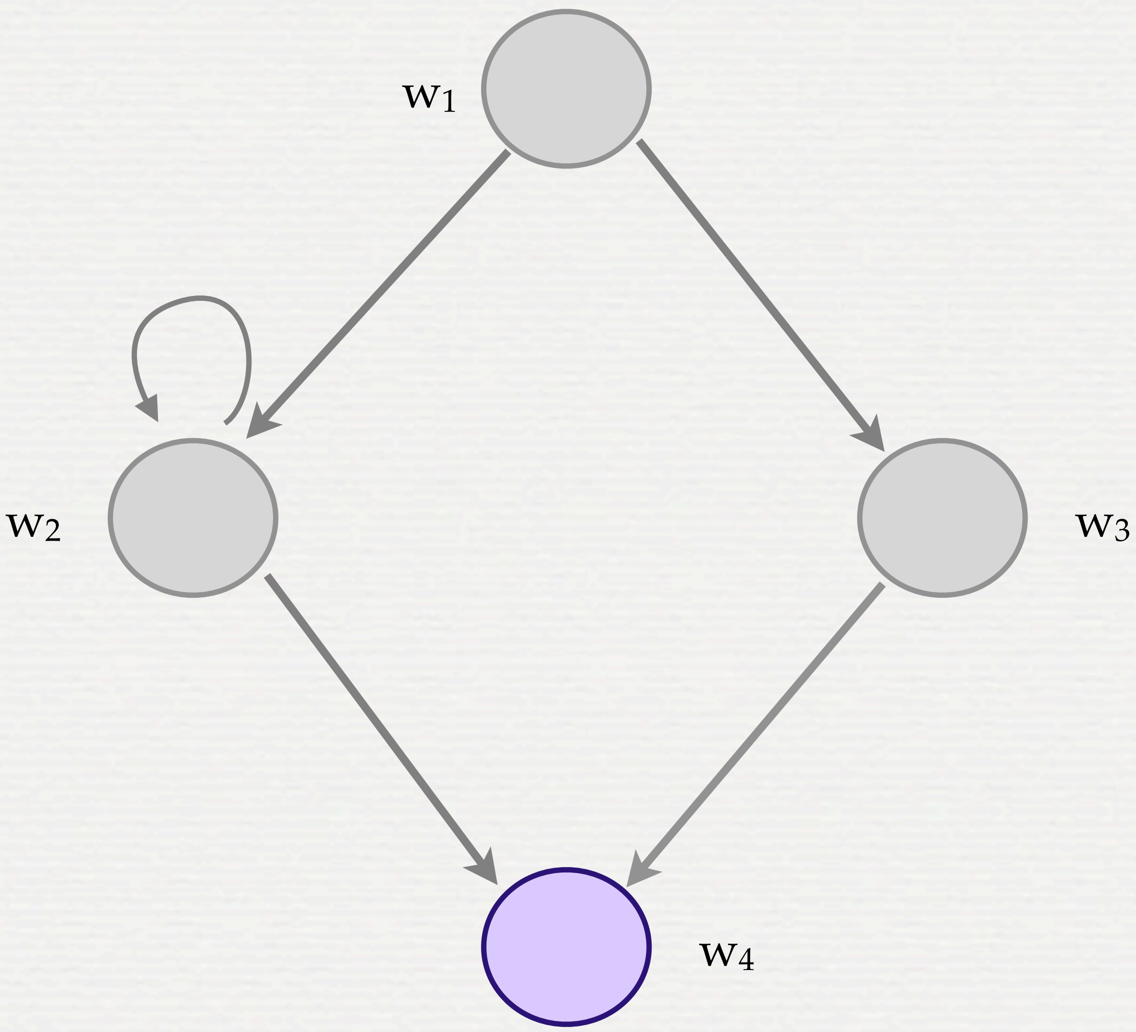


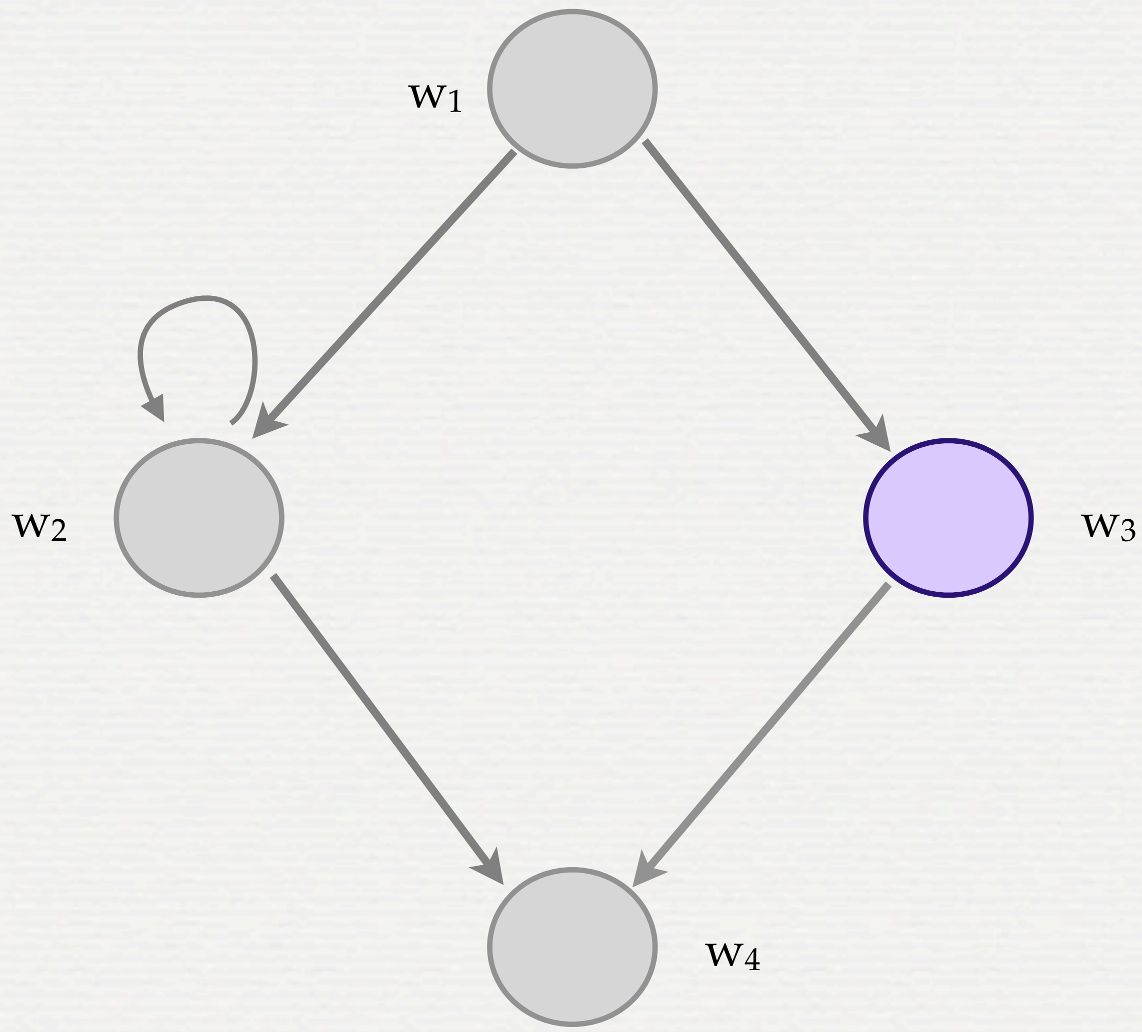
$w_4 \vDash$

DEFINING STATES



$w_4 \vDash \Box\bot$

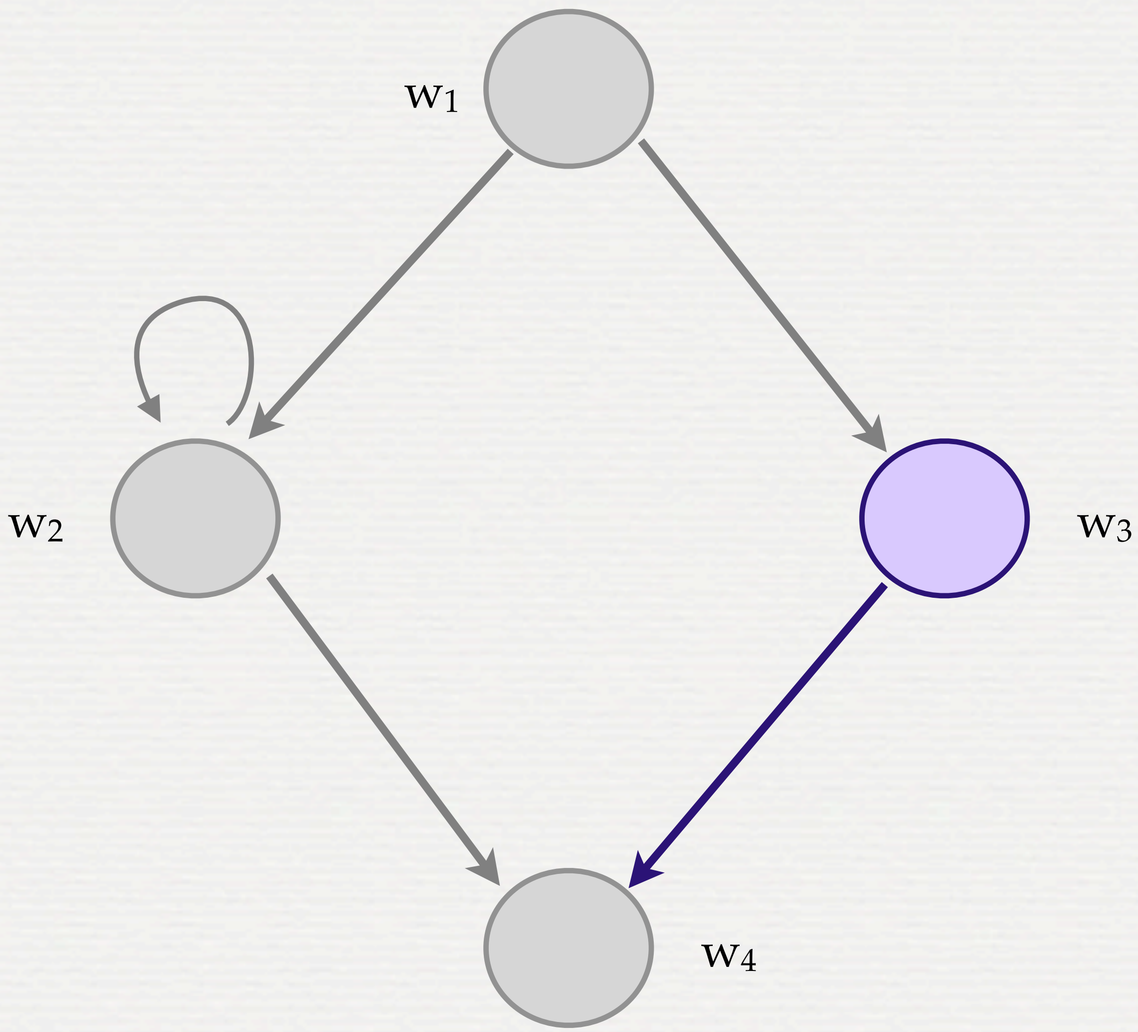
DEFINING STATES



$w_4 \vDash \Box\bot$

$w_3 \vDash$

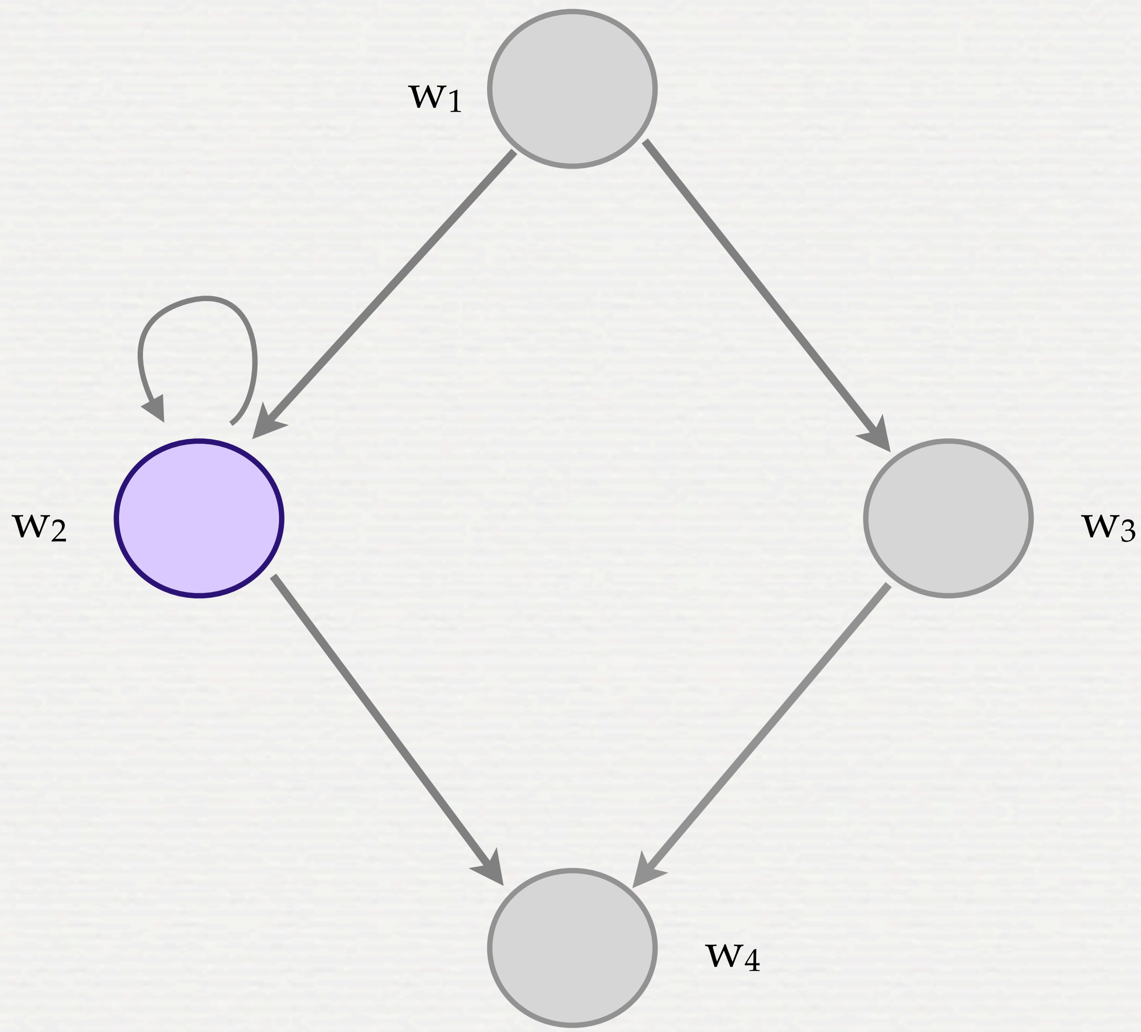
DEFINING STATES



$w_4 \models \Box\bot$

$w_3 \models \Diamond\Box\bot \wedge \Box\Box\bot$

DEFINING STATES

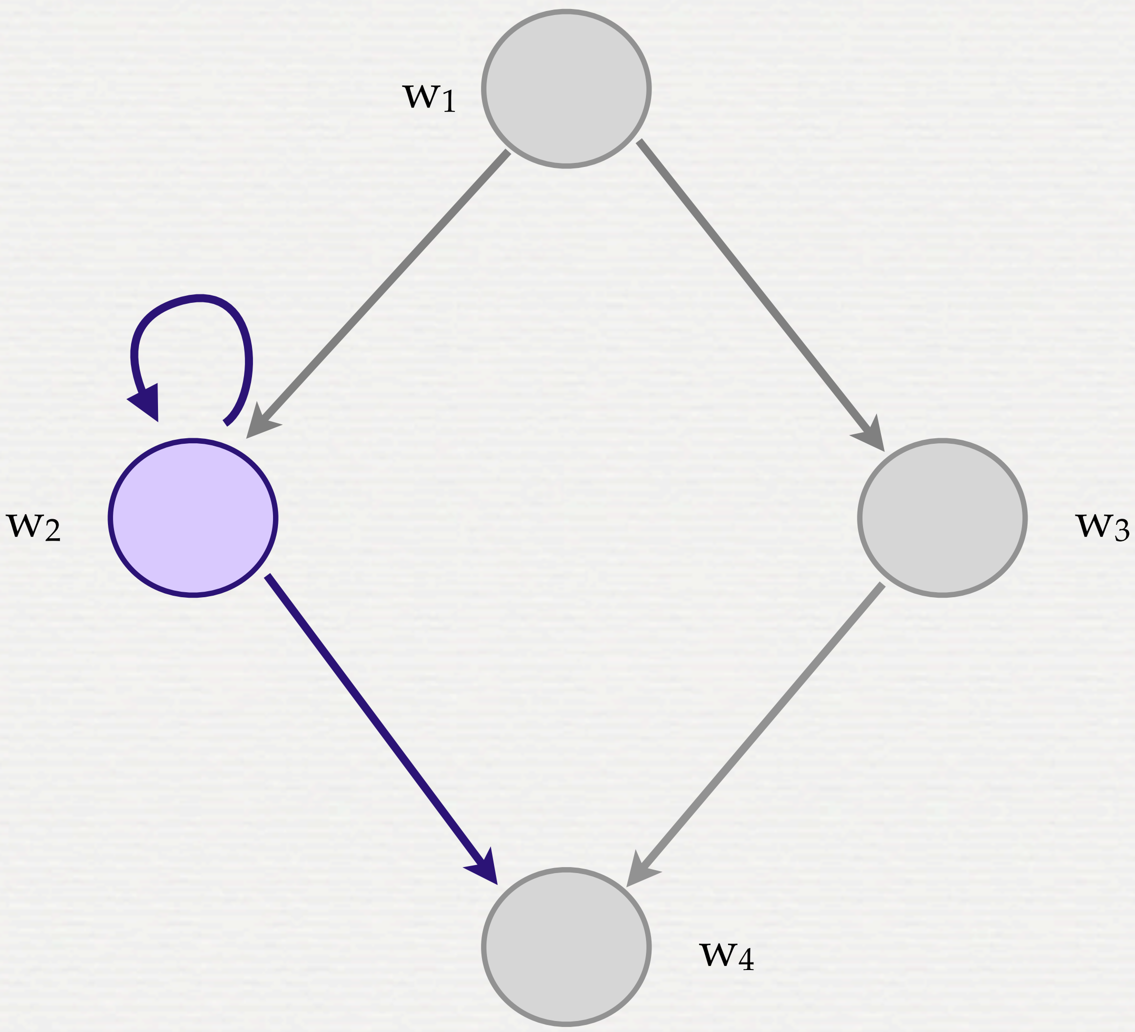


$w_4 \models \Box\bot$

$w_3 \models \Diamond\Box\bot \wedge \Box\Box\bot$

$w_2 \models$

DEFINING STATES

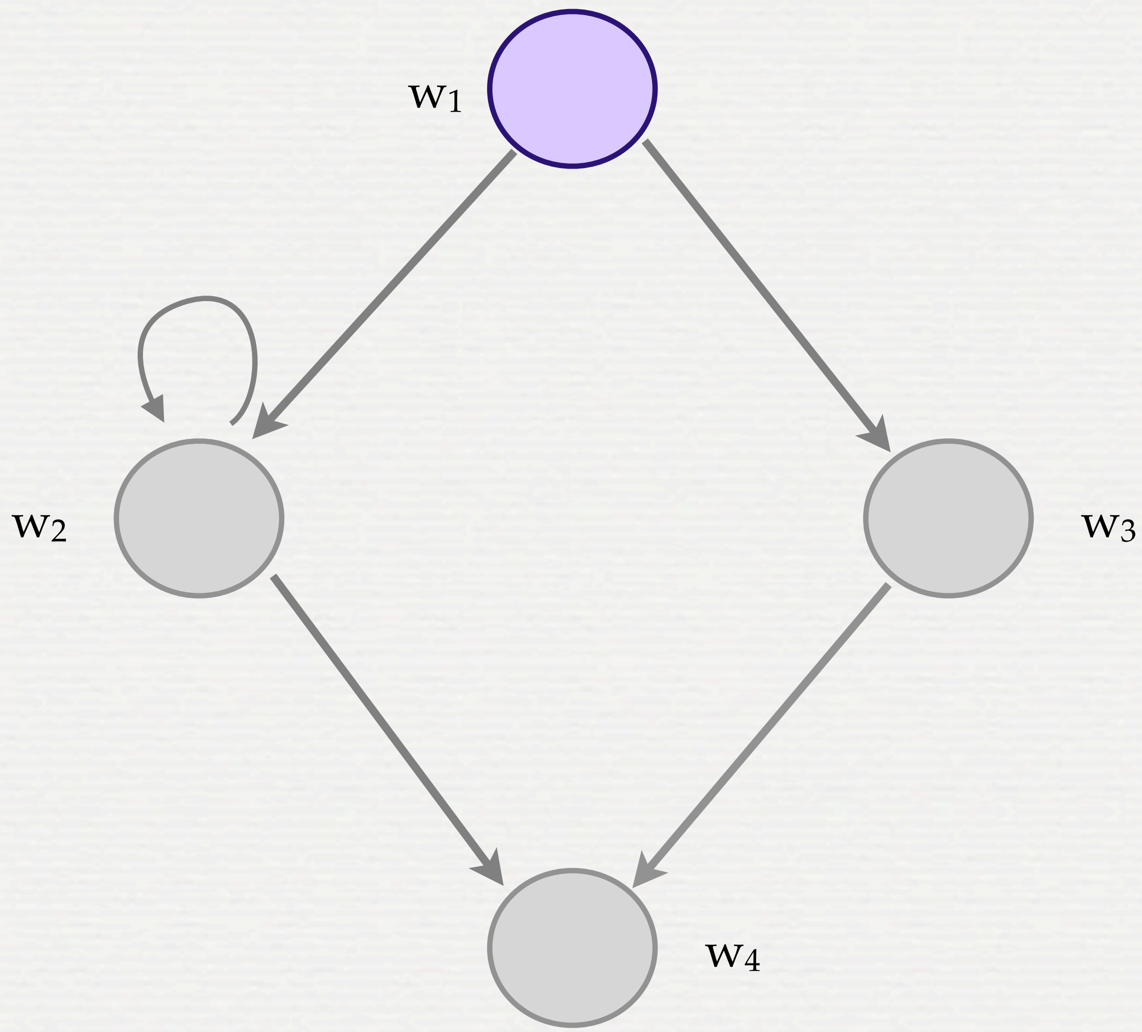


$w_4 \models \Box\bot$

$w_3 \models \Diamond\Box\bot \land \Box\Box\bot$

$w_2 \models \Diamond\Box\bot \land \Diamond\Diamond\top$

DEFINING STATES



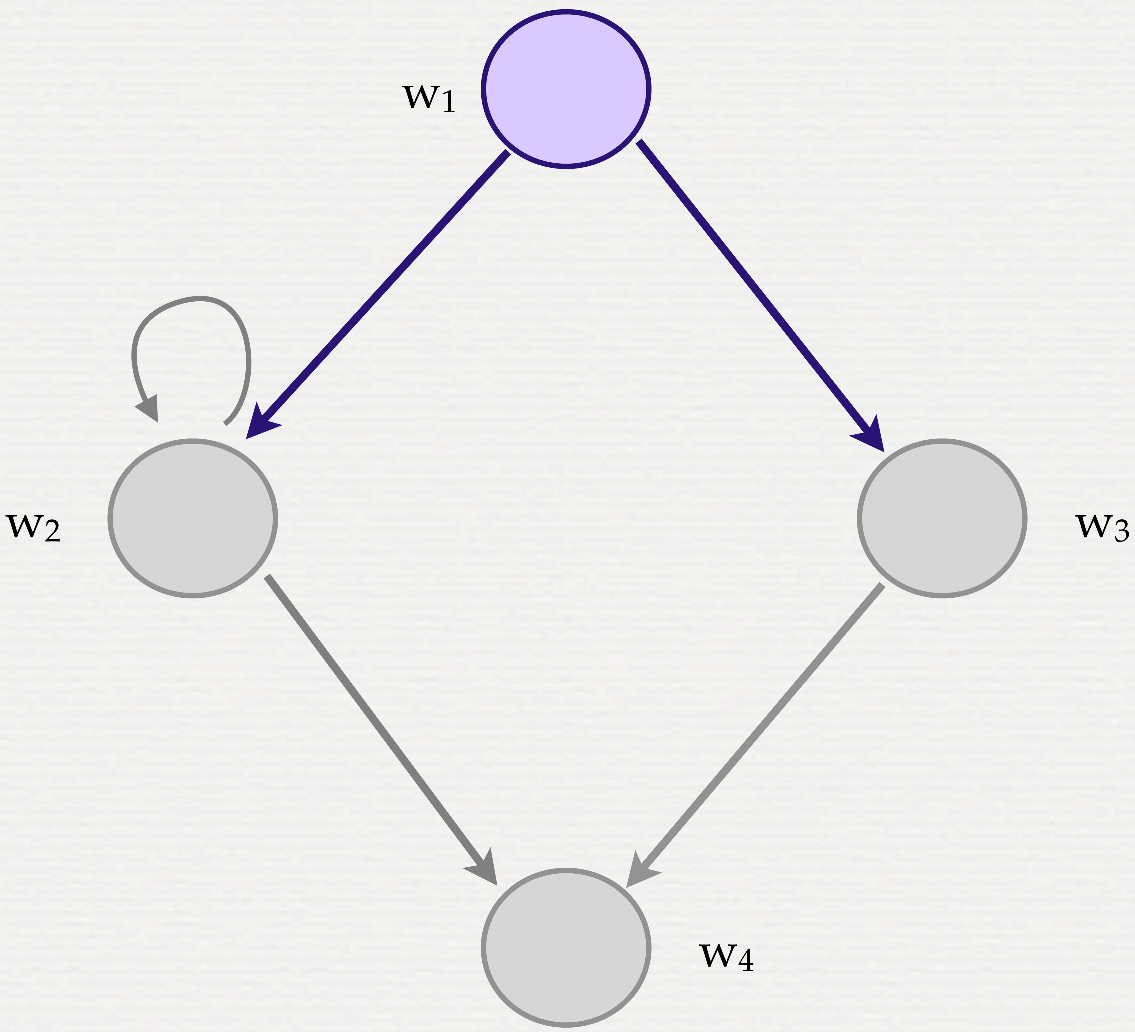
$w_4 \models \Box\bot$

$w_3 \models \Diamond\Box\bot \wedge \Box\Box\bot$

$w_2 \models \Diamond\Box\bot \wedge \Diamond\Diamond\top$

$w_1 \models$

DEFINING STATES



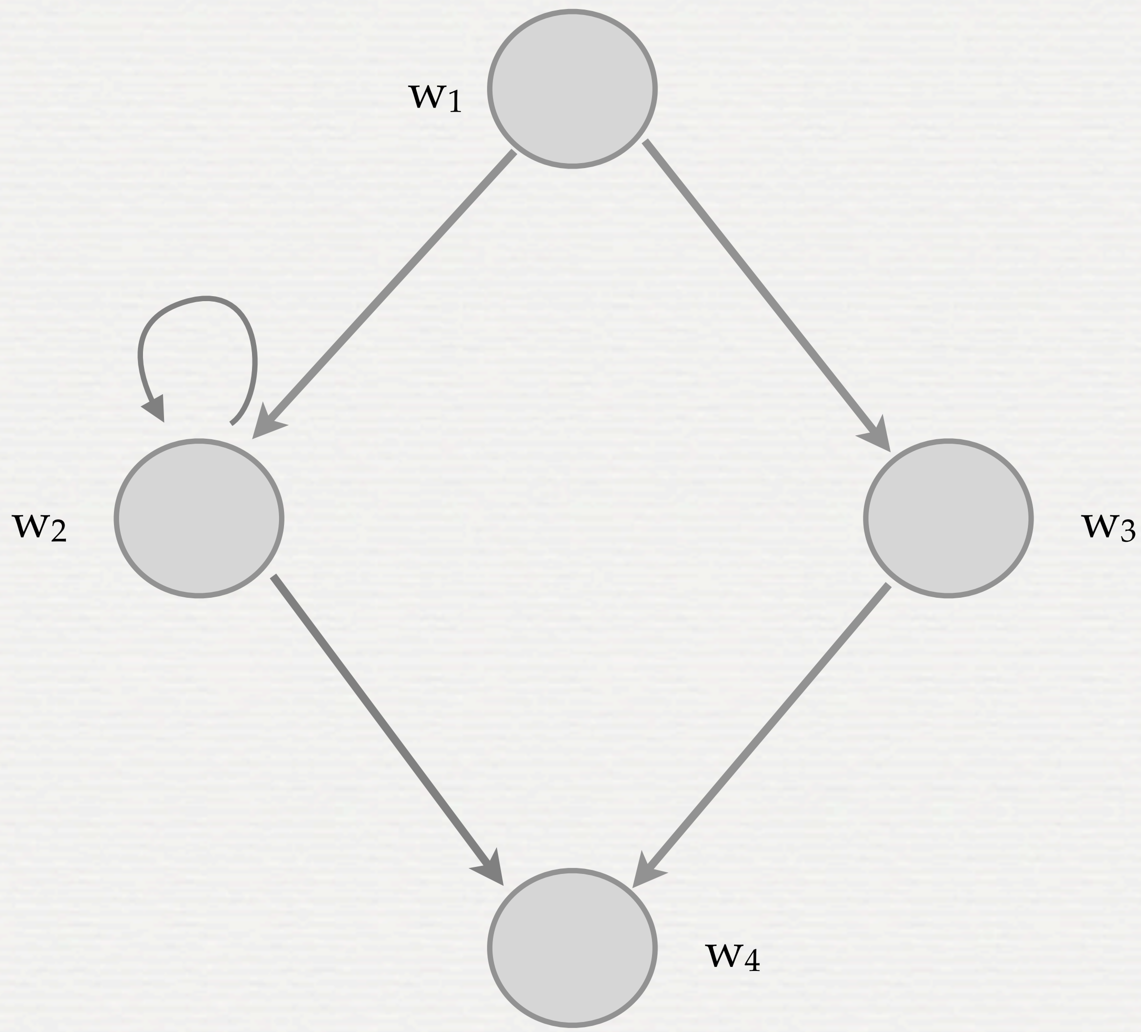
$w_4 \models \Box\bot$

$w_3 \models \Diamond\Box\bot \wedge \Box\Box\bot$

$w_2 \models \Diamond\Box\bot \wedge \Diamond\Diamond\top$

$w_1 \models \Diamond(\Diamond\Box\bot \wedge \Box\Box\bot)$

DEFINING STATES



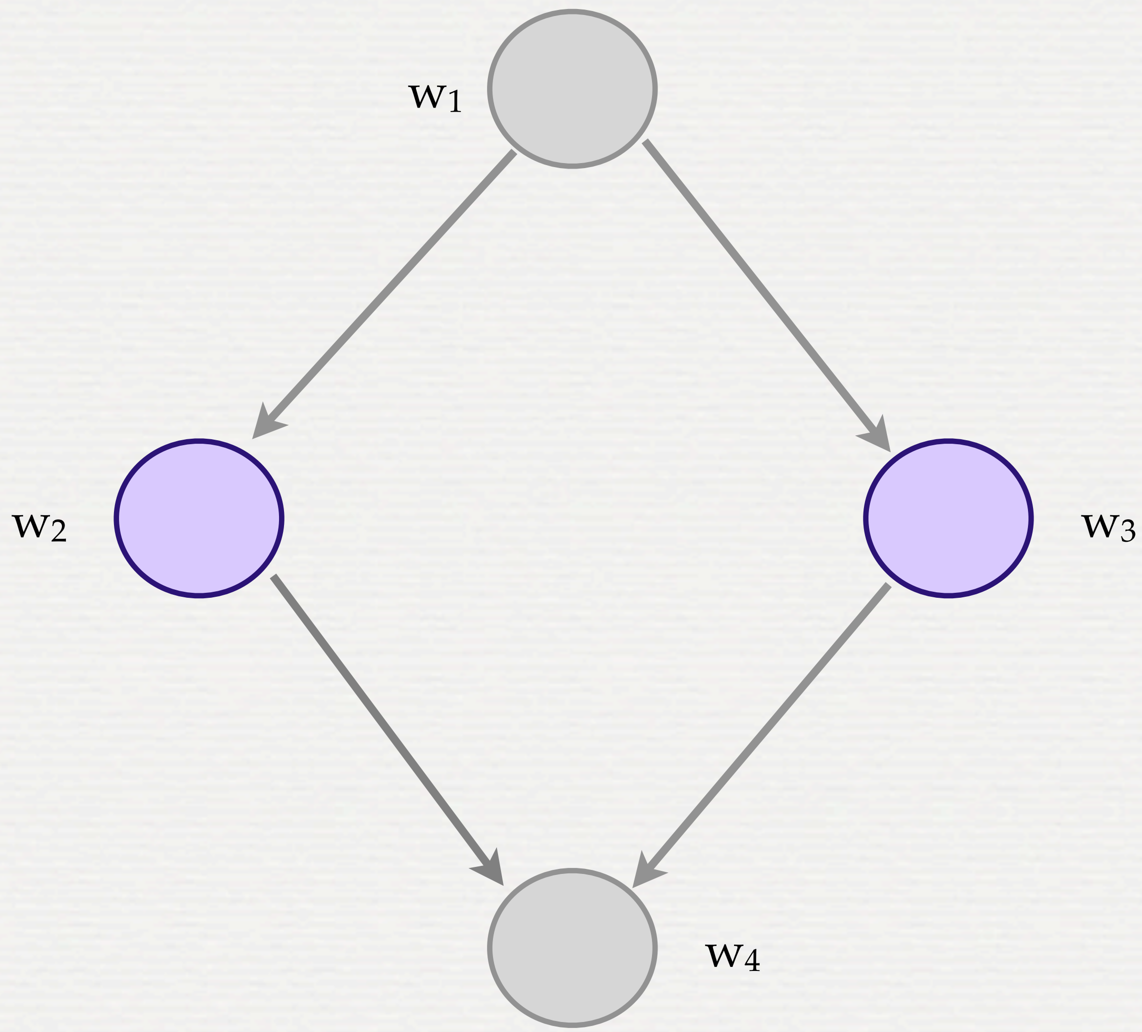
$w_4 \models \Box\bot$

$w_3 \models \Diamond\Box\bot \land \Box\Box\bot$

$w_2 \models \Diamond\Box\bot \land \Diamond\Diamond\top$

$w_1 \models \Diamond(\Diamond\Box\bot \land \Box\Box\bot)$

DEFINING STATES



$w_4 \models \Box\bot$

$w_3 \models \Diamond\Box\bot \wedge \Box\Box\bot$

$w_2 \models \Diamond\Box\bot \wedge \Box\Box\bot$

$w_1 \models \Diamond(\Diamond\Box\bot \wedge \Box\Box\bot)$