LECTURE 18

27.03.2024

Recap of modal logic syntax and semantics

Syntax :

Let $\wp$ be a countable set of propositional variables. Then, formulas of the basic modal logic (BML) is given by :

$$\varphi, \psi := p \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \to \psi \mid \Box\varphi \mid \Diamond\varphi,$$

where, $p \in \wp$.

Semantics :

$\mathcal{M} = (W, R, V)$, where W is a non-empty set of states, $R \subseteq W \times W$ and $V : \wp \to 2^W$.

Truth definition :

Given a modal formula φ, a model $\mathcal{M}$ and a world $w \in W$ in $\mathcal{M}$, φ is satisfied at $\underbrace{(\mathcal{M}, w)}_{\text{pointed model}}$ $[\mathcal{M}, w \models \varphi]$ if : the following holds :

$\mathcal{M}, w \models p$ iff $w \in V(p)$

$\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$

$\mathcal{M}, w \models \varphi \vee \psi$ iff $\mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$

$\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$

$\mathcal{M}, w \models \varphi \to \psi$ iff $\mathcal{M}, w \models \psi$ whenever $\mathcal{M}, w \models \varphi$

$\mathcal{M}, w \models \Box\varphi$ iff for all $v \in W$, wRv implies $\mathcal{M}, v \models \varphi$

$\mathcal{M}, w \models \Diamond\varphi$ iff there exists $v \in W$ s.t. wRv and $\mathcal{M}, v \models \varphi$

We have done a lot of examples in the last class and here we are just recapitulating the definitions.

<u>Satisfiability and Validity of formulas</u>:

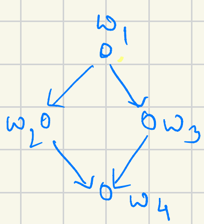
- φ is <u>satisfiable</u> if there is a model $\mathcal{M} = (W, R, V)$ and some $w \in W$ s.t. $\mathcal{M}, w \models \varphi$.
- φ is <u>valid</u> iff its negation is not satisfiable.

H.W. Check whether the following formulas are valid :

- $\Box(p \vee \neg p)$
- $\Box p \vee \Box \neg p$
- $\Box \neg p \to \neg \Box p$
- $\Diamond(\varphi \wedge \psi) \to (\Diamond \varphi \wedge \Diamond \psi)$
- $(\Diamond \varphi \wedge \Diamond \psi) \to \Diamond(\varphi \wedge \psi)$
- $\Box \varphi \to \varphi$
- $\Box \varphi \to \Box \Box \varphi$
- $\neg \Box \varphi \to \Box \neg \Box \varphi$
- $\Box \varphi \to \Diamond \varphi$
- $\Diamond \varphi \leftrightarrow \neg \Box \neg \varphi$

Now, continuing from the last class, let us consider the following model.

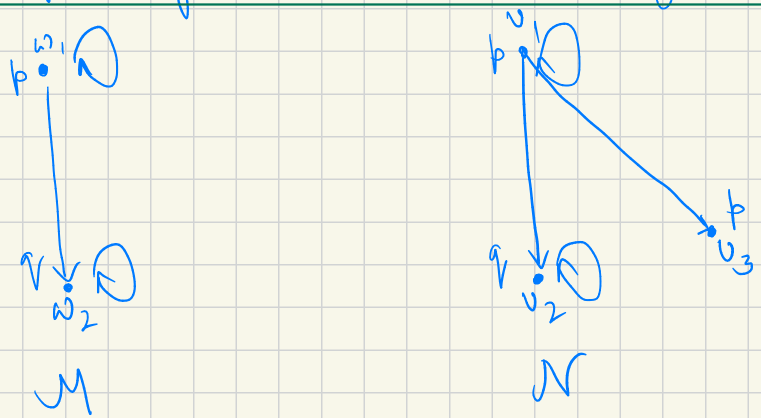
M :



<u>Is it possible to distinguish w_2 and w_3 in this model by modal formulas ?</u>

H.W. Prove that $(\mathcal{M}, w_2)$ and $(\mathcal{M}, w_3)$ satisfy the same modal formulas. In other words, prove that for all modal formulas φ, $\mathcal{M}, w_2 \models \varphi$ iff $\mathcal{M}, w_3 \models \varphi$.

From defining states to distinguishing states.



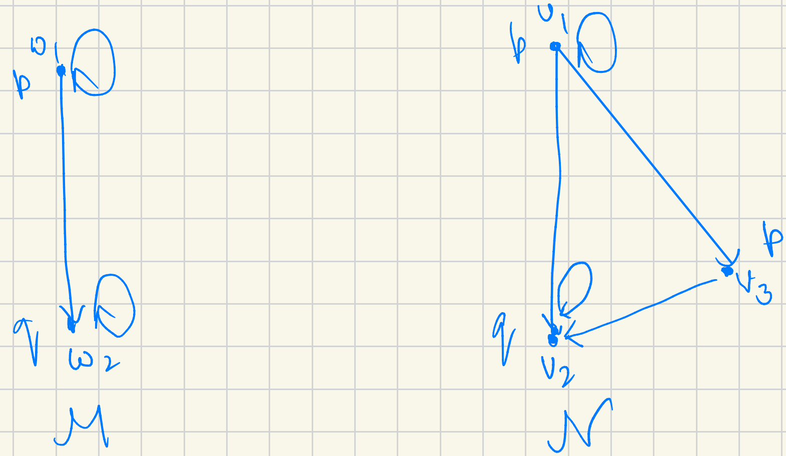
Q. Can you distinguish between the pointed models $(\mathcal{M}, w_1)$ and $(\mathcal{N}, v_1)$ by a modal formula (Informally, can you distinguish between the worlds w_1 in $\mathcal{M}$ and v_1 in $\mathcal{N}$)?

p, $\Diamond q$, $\Box q$, $\Diamond \Box p$

✗ ✗ ✗ ✓

So, $\Diamond\Box p$ holds at $(\mathcal{N}, v_1)$ but does not hold at $(\mathcal{M}, w_1)$. Thus, the formula distinguishes the pointed models.

What about $(\mathcal{M}, w_1)$ and $(\mathcal{N}, v_1)$ now?



In fact, no modal formulas can distinguish between $(\mathcal{M}, w_1)$ and $(\mathcal{N}, v_1)$.

How will you prove this?

Generally, we prove statements regarding formulas by applying induction on the size of formulas. But, how to prove this in terms of these particular models?

Before going into the detailed discussion, let us introduce some concepts.

Modal equivalence

Two pointed models $(\mathcal{M}, w)$ and $(\mathcal{N}, v)$ are said to be modally equivalent if for all modal formulas φ, $\mathcal{M}, w \models \varphi$ iff $\mathcal{N}, v \models \varphi$, that is, $(\mathcal{M}, w)$ and $(\mathcal{N}, v)$ satisfy the same modal formulas.

Bisimulation

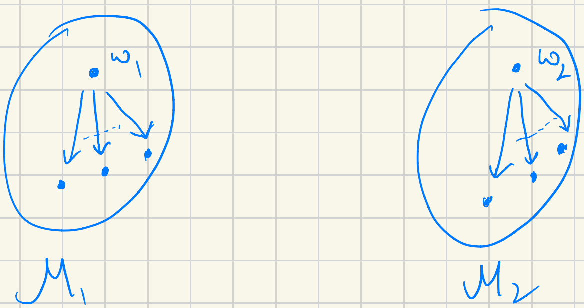
Let $\mathcal{M}_1 : (W_1, R_1, V_1)$ and $\mathcal{M}_2 : (W_2, R_2, V_2)$ be two (Kripke) models. Let $w_1 \in W_1$ and $w_2 \in W_2$. We say that $(\mathcal{M}_1, w_1)$ is bisimilar to $(\mathcal{M}_2, w_2)$ if there is a binary relation $Z \subseteq W_1 \times W_2$, s.t. $w_1 Z w_2$, and for all $x \in W_1$, $y \in W_2$, if $x Z y$, then,

(1) [atomic harmony] for all propositional variables $p \in \mathcal{P}$, $p \in V_1(x)$ iff $p \in V_2(y)$

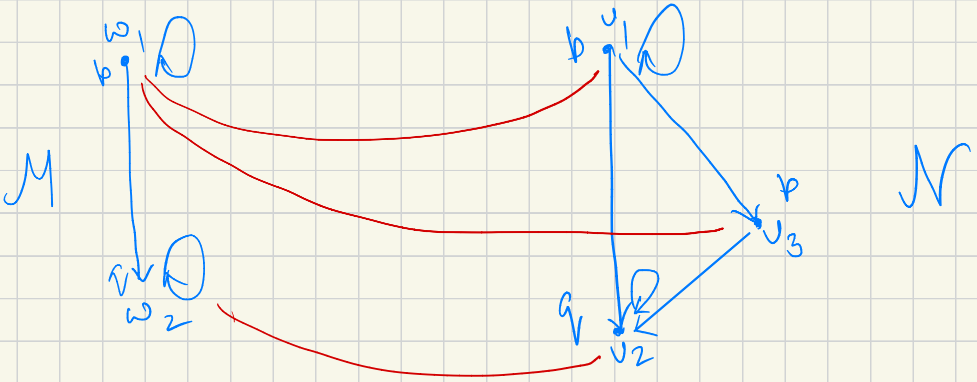
(2) [zig] if $x R_1 x'$ in M_1, then there exists $y' \in W_2$ s.t. $y R_2 y'$ in M_2 and $x' Z y'$.

(3) [zag] if $y R_2 y'$ in M_2, then there exists $x' \in W_1$ s.t. $x R_1 x'$ in M_1 and $x' Z y'$.

We write $M_1, w_1 \simeq M_2, w_2$.



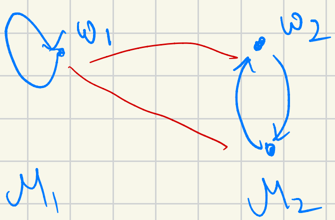
Let's refer back to the example we had before.

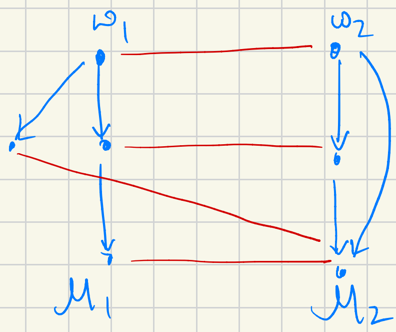


$$Z = \{(w_1, v_1), (w_2, v_2), (w_1, v_3)\}$$

We have: $\mathcal{M}, w_1 \cong \mathcal{N}, v_1$

More examples



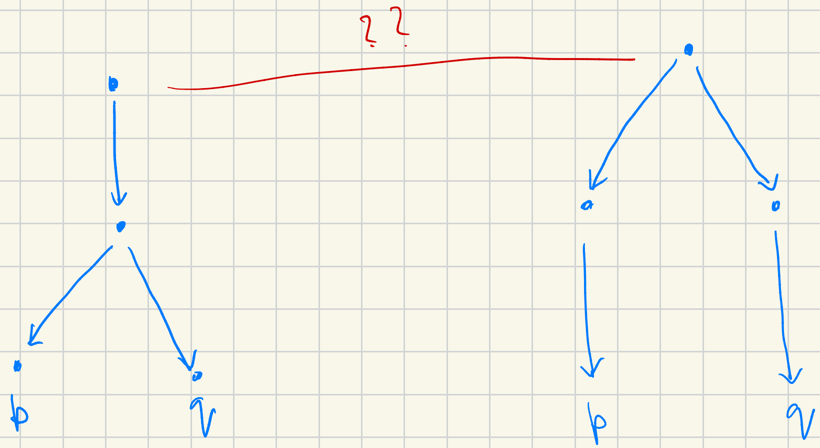


$\mathcal{M}_1, w_1 \cong \mathcal{M}_2, w_2$

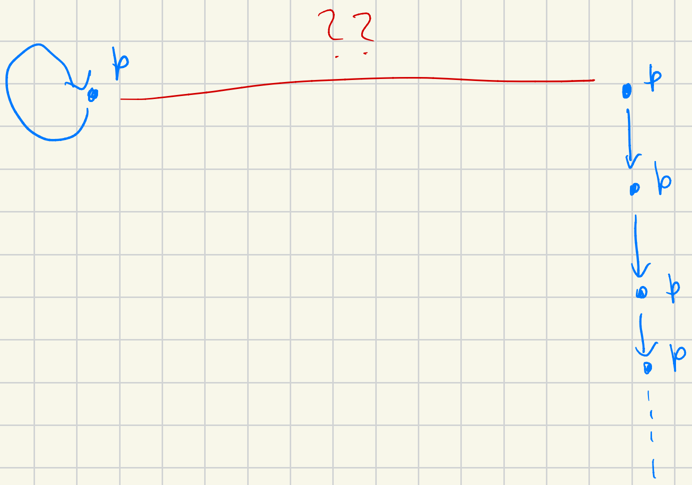
$\mathcal{M}_1, w_1 \cong \mathcal{M}_2, w_2$

H.W.

1.



2.



Connecting the notions of 'modal equivalence' and 'bisimulation'.

<u>Invariance Lemma:</u>

Let $(\mathcal{M}, s)$ and $(\mathcal{N}, t)$ be two pointed models. If $(\mathcal{M}, s) \cong (\mathcal{N}, t)$ [$(\mathcal{M}, s)$ is bisimilar to $(\mathcal{N}, t)$], then $(\mathcal{M}, s)$ and $(\mathcal{N}, t)$ are modally equivalent.

Proof: Let $(\mathcal{M}, s) \cong (\mathcal{N}, t)$. To prove that for all modal formulas φ, $(\mathcal{M}, s) \models \varphi$ iff $(\mathcal{N}, t) \models \varphi$. We prove this by applying induction on the size of a formula φ.

Base case: Let $\varphi = p$. Then the result holds by Condition (1) of the definition of bisimulation.

Induction Hypothesis: Suppose the result holds for all formulas of size $\leq n$.

Induction Step: Let the size of φ be $n+1$. Then we have the following cases:

Case 1: $\varphi := \neg\psi$. Then $\mathcal{M}, s \models \varphi$ iff $\mathcal{M}, s \models \neg\psi$ iff $\mathcal{M}, s \not\models \psi$ iff $\mathcal{N}, t \not\models \psi$ (by I.H.) iff $\mathcal{N}, t \models \neg\psi$ iff $\mathcal{N}, t \models \varphi$.

Case 2: $\varphi := \psi \vee \chi$. Then, $\mathcal{M}, s \models \varphi$ iff $\mathcal{M}, s \models \psi \vee \chi$ iff $\mathcal{M}, s \models \psi$ or, $\mathcal{M}, s \models \chi$ iff $\mathcal{N}, t \models \psi$ or $\mathcal{N}, t \models \chi$ (by I.H.) iff $\mathcal{N}, t \models \psi \vee \chi$.

Case 3: $\varphi := \Diamond\psi$. Suppose $\mathcal{M}, s \models \varphi$, i.e. $\mathcal{M}, s \models \Diamond\psi$. Then, there is s' in $\mathcal{M}$, s.t. $s R_{\mathcal{M}} s'$ and $\mathcal{M}, s' \models \psi$. Now, by the zig-condition, since $\mathcal{M}, s \simeq \mathcal{N}, t$, there is a t' in $\mathcal{N}$ s.t. $t R_{\mathcal{N}} t'$ and $\mathcal{M}, s' \simeq \mathcal{N}, t'$. So, by I.H., $\mathcal{N}, t' \models \psi$.

Hence, $\mathcal{N}, t \models \Diamond\psi$. So we have:

$\mathcal{M}, s \models \Diamond\psi$ implies $\mathcal{N}, t \models \Diamond\psi$. The other direction can be proved similarly using the Zag-condition.

This completes the proof.

What about the converse?

If two pointed model $(\mathcal{M}, s)$ and $(\mathcal{N}, t)$ satisfy the same modal formulas, are they bisimilar?

NO !!

H.W Prove that $(\mathcal{M}, s)$ and $(\mathcal{N}, t)$ are modally equivalent but $(\mathcal{M}, s) \not\leftrightarrow (\mathcal{N}, t)$.