Lecture 18

1 Sis inner Lation Let  $M_1$ :  $(W_1, R_1, V_1)$  and  $M_2$ :  $(W_2, R_1, V_2)$  be two models. Act w, EW1 and w2 EW2. We say that (M, w,) is bisini les to  $(\mathcal{M}_1, \omega_2)$   $((\mathcal{M}_1, \omega_1) \simeq (\mathcal{M}_2, \omega_2))$  if there is a binary relation Z C W, X W2 such that  $w_1 Z w_2$  and for all  $r \in W_1, y \in W_2$ , if a try, then : 1. A tomic harmony: for all propositional veriable  $p \in P$ ,  $n \in V_1(p)$  iff  $Y \in V_2(p)$ 2. Zig if 2R, X, then there enists y E W2 such that y R2 y and 2 Zy. 3. Zag: if  $y k_2 y'$ , then there exists  $x' \in W$ , such that  $x k_1 x'$  and x' Z y'. Example :





Induction by pothesis Suppose the result holds for all formulas of ruge < m Induction Slep: Let the size of q be m+1. Case 1: q:= 7 y. M, S F 7 y iff M, S F y if  $N, t \neq \psi(by I.H.)$  if  $(N, t) \neq \gamma \psi$ . Case 2: Q'= VX - Jollows Semi larly Case 3: Q'= QY. Suppose M, SE QY. Then, there is s' such that s Ry s' and  $M, s' \neq \Psi$ . Now, by the zig-cond-ition, pince  $(M, s) \simeq (N, t)$ , there is t' in N such that t RN t' and (M, s')  $\simeq (N, t')$ . Thun, by I.H., N, t'=  $\psi$ . Ihm N, t E QY. So, if M, S F QY then N, t F Qy. The other direction can be shown similarly using Zag - condition. This completes the proof. H.W. Show that DQ <> 1070 is a valid

What about the converse? If two from ted models (M, S) and (N, t) satisfy the same modal formulas. are they bis michan? NO 1 × × × × × M H.W. Prove that (M, 5) and (N, t) are modelly equivalent but are not bisi milar. Model depth of a formula: Given a model formula q, the model depth of q (md (a)) is defined induc tive by as follows md(p)=0; md(ry)=md(ry);  $md(qvy)-md(qny)=md(q \rightarrow y)=md(q \rightarrow y)$ = max {md (q), md (4)} md(Dq) = md(Qq) = md(q) + 1

So we see that the converse result (mo dal equivalence implies bisimu lation) may not always hold. Com we pind a restricted class of models where the converse does hold ? het us consider finite models and check whether the result holds. Let M and N be finite -models with (M, S) and (N, t) being modelly equivalent. Are they bisimilar? Jo show that  $(M, s) \simeq (N, t)$ , we need to find a bisimulation  $Z \subseteq$  $W_{M} \times W_{N}$  with s Z t. Let us consider Z C Wy X WN defined as follows: for all n E Wy and y E Wy, a Zy iff x and y satisfy the same model formules Of course, sZt

We just need to show that Z is a bisi mulation. het n E WM and y E Wr Jelt n Zy 1. Condition (AM) follows 2. Let us prove (Zig) now. Suppose n Ryn To show that there exists y E WN such that y RN y and n'Zy' Suppose not. Then, There does not exist any y in Wy such that y Rny and nZy het T= {u: yRNuZ. Cen T be empty? No. Since M, 2 F QT and 2 Zy, po, (N, y) = OT, Ihus, T is non-emply Since N is finite, T is also finite

det T= Eu, uz, --.., um3. Ihen, for all i=1,-., m, it is not the case that x' Z ui ' Then, for each i, there is a modal formula qi, say, such that M, n = q; and N, u; F q; So,  $M, \chi' \neq Q, \Lambda Q_2 \Lambda - - \Lambda Q_n$ , whereas  $N, n; \notin Q; for each i. So, we have:$  $M, n \not\models \Diamond (q_1 \land q_2 \land \dots \land q_n), and$  $\mathcal{N}, \mathcal{Y} \neq \mathcal{O}(\mathcal{Q}_1 \wedge \mathcal{Q}_2 \wedge \cdots \wedge \mathcal{Q}_n)$ So, we have a contradiction as 2 Z y. This completes the proof of the (Zig) - condition 3. Condition (Zag) can be proved sim Marly. This completes the proof 

So, for finite models, model equiv alence implies bis indetron Ohere did we use the finite condition? We used the fact that T is finite Image - finste models. A Kripke mode M= (W, R, V) is said to be image - finite if for all w E W, the set Iw= {w E W wRw/J is finite. So, instead of finite models if we consider image-finite models, the converse result still goes through So we have the following result : Henessy-Milner Theorem.

het M and N be two image finite models. Then (M, S) and (N, t) are modally equivalent iff they are bisimilar An application on expressivily We use this concept to show what model logic can/cannot express Universal operator U M, w = Uq iff for all v E WM.  $\mathcal{M}, \mathcal{O} \models \mathcal{Q}$ Is the universal of erator definable in the model language? Suppose it is. Let X(q) be a model formula such that for all models M and for all wim M,

M, w F U q iff  $\mathcal{M}, w \not\models \mathcal{A}(q)$ the following models Now, consider ω **,** μ M: M, w F Up but, M, w'H Up So, M,  $\omega \neq \alpha(p)$  and, M',  $\omega' \neq \alpha(p)$ plowever, (M, w) and (M', w) are Jusi mitar, hence sati sfy the same model formulas. So, we should have had.,  $M, w \not\models x(p)$  iff  $M', w' \not\models x(p)$ . But, that is not the case, so we have a contradiction Ilus, the universal operator U is not expressible in the logic PML.