LECTURE 19

01.04.2024

So, we have that bisimulation implies modal equivalence, but the converse may not be true.

- What condition can we put on the models to make the converse true?

We can consider finite models. May be, we can go a little beyond that. Let's see.

Proposition: Let M and N be two finite models. Suppose $(\mathcal{M}, s)$ and $(\mathcal{N}, t)$ are modally equivalent. Then, they are bisimilar.

Proof: Let $(\mathcal{M}, s)$ and $(\mathcal{N}, t)$ be two pointed models satisfying the same modal formulas. To show that they are bisimilar. Consider the binary relation Z between W_M and W_N defined by: xZy iff $(\mathcal{M}, x)$ and $(\mathcal{N}, y)$ are modally equivalent. So, we now need to show that Z is a bisimulation.

Let $x \in W_M$ and $y \in W_N$, s.t. $x Z y$.

(i) Z satisfies the condition of Atomic Harmony.

(ii) Let us now consider the Zig condition.

We have $x Z y$. Let $x' \in W_M$ s.t. $x R_M x'$.

To show that there exists $y' \in W_N$ s.t. $y R_N y'$ and $x' Z y'$. Suppose not. Then, there does not exist any $y' \in W_N$ s.t. $y R_N y'$ and $x' Z y'$. Let $T = \{u : y R_N u\}$. Can T be empty? No. Why? If T is empty, then $N, y \models \Box \bot$. Then, $M, x \models \Box \bot$, as $x Z y$. But, this is a contradiction as there is x' in W_M s.t. $x R_M x'$. So, T is non-empty. Now as N is finite, T is finite. Let $T = \{u_1, u_2, \ldots, u_m\}$. Now, we have that there exist no u in W_N, s.t. $y R_N u$ and $x' Z u$. Then, for all $i = 1, 2, \ldots, m$, it is not the case that $x' Z u_i$. Then, for each i, there is a modal formula φ_i, say, s.t. $\mathcal{M}, x' \models \varphi_i$ and $\mathcal{N}, u_i \not\models \varphi_i$.

So, $\mathcal{M}, x' \models \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_m$ and $\mathcal{N}, u_i \not\models \varphi_i$ for each i. So, we can conclude that

$\mathcal{M}, x \models \Diamond(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_m)$ and,

$\mathcal{N}, y \not\models \Diamond(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_m)$ Check!

So, we have a contradiction, as $(\mathcal{M}, x)$ and $(\mathcal{N}, y)$ satisfy the same modal formulas.

Hence, our original assumption was wrong and we have proved the Zig condition.

(iii) Condition Zag can be proved similarly.

This completes the proof.

Note: Let us now introduce the notion of image-finite models - The Kripke models where each world is related to only finitely many worlds. The above proposition would continue to hold for image-finite models.

The corresponding theorem is called:

Hennessy - Milner theorem.

One of the main applications of the bisimulation concept is to show how expressive this basic modal logic is:

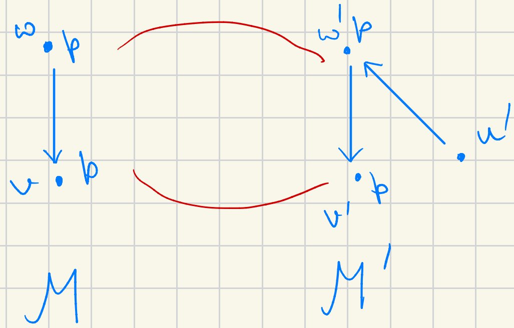
Example: Universal operator ($\mathcal{U}$)

$\mathcal{M}, w \models \mathcal{U}\varphi$ iff for all $v \in W_{\mathcal{M}}$, $\mathcal{M}, v \models \varphi$.

Is this universal operator $\mathcal{U}$ definable in our basic modal language?

Answer: Suppose it is. Let $\alpha(\varphi)$ be a modal formula s.t. for all models $\mathcal{M}$ and all worlds w in $\mathcal{M}$,

$$\mathcal{M}, w \models \mathcal{U}\varphi \quad \text{iff} \quad \mathcal{M}, w \models \alpha(\varphi).$$



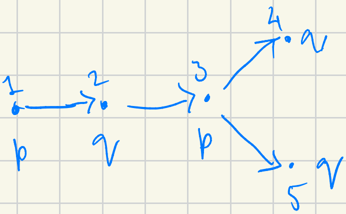
$\mathcal{M}, w \models \mathsf{U} p$ $\qquad\qquad$ $\mathcal{M}', w' \not\models \mathsf{U} p$.

$\mathcal{M}, w \models \alpha(p)$ $\qquad\qquad$ $\mathcal{M}', w' \not\models \alpha(p)$

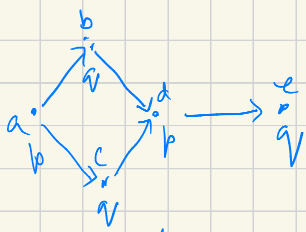
Moreover, $(\mathcal{M}, w)$ and $(\mathcal{M}', w')$ are bisimilar. So, $\mathcal{M}, w \models \alpha(p)$ iff $\mathcal{M}', w' \models \alpha(p)$. But that is not the case here. So, we have a contradiction. Thus, the universal operator U is not definable / expressible in basic modal logic.

First-order logic vs. Basic modal logic

How similar / different they are?



$\mathcal{M}$



$\mathcal{M}'$

$$(\mathcal{M}, 1) \leftrightarrows (\mathcal{M}', a)$$

bisimilar

(modally - equivalent)

An FOL formula

$$\exists y_1 \exists y_2 \exists y_3 \left(y_1 \neq y_2 \wedge y_1 \neq y_3 \wedge y_2 \neq y_3 \wedge Rxy_1 \wedge Rxy_2 \wedge Ry_1y_3 \wedge Ry_2y_3 \right)$$

So, modal logic cannot distinguish between these pointed models, but first-order logic can.

What is the precise relationship between FOL and ML?

To consider this, we first need to decide on the language of FOL to deal with BML.

Parameters of the language

R (a binary predicate symbol)

$P_0, P_1, P_2, \ldots$ (unary predicate symbols)

Then, a Kripke model, (W, R, V) can be seen as a first-order structure (D, I):

$$D = W$$
$$I(R) = R \subseteq W \times W$$
$$I(P_i) = V(p_i) \subseteq W \quad [V: \mathcal{P} \to 2^W].$$

A translation of BML formulas into FOL formulas:

We consider a variable x for this translation, known as the 'Standard Translation' ST_x:

$$ST_x(p_i) = P_i x$$
$$ST_x(\bot) = \neg(x = x)$$
$$ST_x(\neg\varphi) = \neg ST_x(\varphi)$$
$$ST_x(\varphi \vee \psi) = ST_x(\varphi) \vee ST_x(\psi)$$
$$ST_x(\Box\varphi) = \forall y (Rxy \to ST_y(\varphi)).$$
$$ST_x(\Diamond\varphi) = \exists y (Rxy \wedge ST_y(\varphi))$$

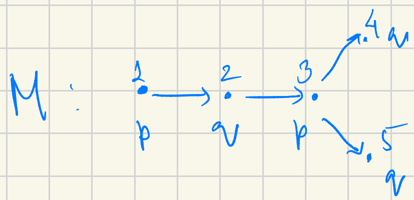
- Find: $ST_x(\Diamond(\Box p \vee q))$

$ST_x(\Diamond(\Box p \vee q))$

$= \exists y\,(Rxy \wedge (\forall z\,(Ryz \to Pz) \vee Qy))$

Let us now look at the models / structures for $\mathcal{L}$

Examples



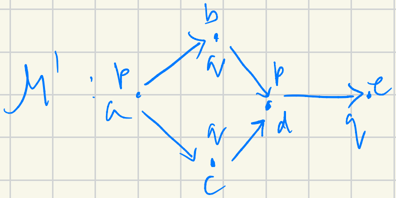
L - structure (D, I).

$D = \{1, 2, 3, 4, 5\}$

$I(P) = V(p) = \{1, 3\}$

$I(Q) = V(q) = \{2, 4, 5\}$.

$I(R) = \{(1,2), (2,3), (3,4), (3,5)\}$



L - structure (D, I)

$D = \{a, b, c, d, e\}$

$I(p) = \{a, d\}$

$I(q) = \{b, c, e\}$

$I(R) = \{(a,b), (a,c), (b,d), (c,d), (d,e)\}$

Any Kripke model can be seen as a first-order structure, that is, an L-structure, where L is given as above.

In the following, let $\Vdash$ denote the modal satisfaction relation and $\vDash$ denote the first order satisfaction relation.

Now, we are ready to prove the following results:

1. For all Kripke models $\mathcal{M}$ and for all worlds w in $\mathcal{M}$: $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}_{[x \to w]} \vDash ST_x(\varphi)$, for all modal formulas φ.

2. For all Kripke models $\mathcal{M}$, $\mathcal{M} \Vdash \varphi$ ($\mathcal{M}, w \Vdash \varphi$ for all w in $\mathcal{M}$) iff $\mathcal{M} \vDash \forall x\, ST_x(\varphi)$, for all modal formulas φ.

[H.W.]

- We will prove (1) in the next class.