Lecture 2

First-order logic : Semantics

D : a non-empty set (domain)

I : interprete the parameters of the language

$I(c_i) \in D$, $c_i \in \mathcal{C}$

$I(f_i^n) : \underbrace{D \times \cdots \times D}_{n \text{ times}} \to D$

$I(p_i^n) \subseteq \underbrace{D \times \cdots \times D}_{n \text{ times}}$.

$\boxed{(D, I) : \text{a structure}}$.

Groups

$\mathcal{C} = \{c\}$, $\mathcal{F} = \{f_1^2, f_2^1\}$, $\mathcal{P} = \emptyset$, '$=$'

$D = G$

$I(c) = e_G$.

$I(f_1^2) : G \times G \to G$

$I(f_2^1) : G \to G$.

Numbers

$\mathcal{C} = \{c\}$, $\mathcal{F} = \{f_1^1, f_2^2, f_3^2\}$, $\mathcal{P} = \{p_1^2\}$, '$=$'

$D = \mathbb{N}$

$I(c) = 0$

$I(f_1^1) = S$.

$I(f_2^2) = +$

$I(f_3^2) = \cdot$

$I(p_1^2) = <$

Now, we need to give specific meaning to the variables in the domain D. We do that by considering an assignment function,

$g : V \to D$. Now, we have :

(D, I, g) : model

Claim : Any $g : V \to D$ can be extended uniquely to a function $g' : \mathcal{T} \to D$ ($\mathcal{T}$: set of terms)

Proof idea: $g': J \to D$

$$g'(x_i) = g(x_i)$$

$$g'(c_i) = I(c_i)$$

$$g'(f_i^n t_1 t_2 \cdots t_n) = I(f_i^n)(g'(t_1), \cdots, g'(t_n))$$

H.W. Prove the above claim in details. ■

We will denote g' by g from now on.

We are now ready to define:

$$\underbrace{(D, I, g)}_{\text{model}} \underbrace{\models}_{\text{satisfies}} \underbrace{\varphi}_{\text{formula}}$$

1. $(D, I, g) \models t_1 \equiv t_2$ if $g(t_1) = g(t_2)$
2. $(D, I, g) \models p_i^n t_1 \cdots t_n$ if $(g(t_1), \cdots, g(t_n)) \in I(p_i^n)$
3. $(D, I, g) \models \neg\varphi$ if it is not the case that $(D, I, g) \models \varphi$.
4. $(D, I, g) \models \varphi \vee \psi$ if $(D, I, g) \models \varphi$ or $(D, I, g) \models \psi$.

5. $(D, I, \ell) \models \varphi \wedge \psi$ if $(D, I, \ell) \models \varphi$ and $(D, I, \ell) \models \psi$

6. $(D, I, \ell) \models \varphi \to \psi$ if $(D, I, \ell) \models \varphi$ implies $(D, I, \ell) \models \psi$.

7. $(D, I, \ell) \models \varphi \leftrightarrow \psi$ if $(D, I, \ell) \models \varphi$ iff $(D, I, \ell) \models \psi$

8. $(D, I, \ell) \models \forall x \varphi$ if for all $d \in D$, $(D, I, \ell_{[x \to d]}) \models \varphi$

9. $(D, I, \ell) \models \exists x \varphi$ if there exists $d \in D$, such that $(D, I, \ell_{[x \to d]}) \models \varphi$.

How do we define $\ell_{[x \to d]}$?

$\ell_{[x \to d]} : V \to D$, defined by

$$\ell_{[x \to d]}(y) = \begin{cases} d & \text{if } y = x \text{ ; } \ell(y) \text{ if } y \neq x \\ \ell(y) & \text{if } y \neq x \end{cases}$$

$M : (D, I, \ell)$; $M_{[x \to d]} : (D, I, \ell_{[x \to d]})$

H.W. Prove the following:

- For any model $\mathcal{M}$, and any formulas φ, ψ,

(a) $\mathcal{M} \models \forall x \varphi \Leftrightarrow \neg \exists x (\neg \varphi)$

(b) $\mathcal{M} \models \varphi \to \psi$ iff $\mathcal{M} \not\models \varphi$ or $\mathcal{M} \models \psi$.

(c) $\mathcal{M} \models \varphi \vee \neg \varphi$

(d) $\mathcal{M} \models \varphi \vee \psi$ iff $\mathcal{M} \models \neg(\neg \varphi \wedge \neg \psi)$

(e) $\mathcal{M} \models \varphi \Leftrightarrow \psi$ iff $\mathcal{M} \models (\varphi \to \psi) \wedge (\psi \to \varphi)$

In the next class we start with the concepts of free and bound variables.