LECTURE 20

03.04.2024

Proof of 1

We prove this by induction on the size of the formula φ

Base case: $\varphi := p$. Then, $\mathcal{M}, w \Vdash p$ iff $w \in V(p)$ iff $w \in I(P)$ iff $\mathcal{M}_{[x \to w]} \models Px$ iff $\mathcal{M}_{[x \to w]} \models ST_x(p)$.

I.H.: Suppose the result holds for all formulas of size $\leq n$.

I.S.: Suppose φ has size $n+1$.

Case I: $\varphi := \neg\psi$. Then, $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}, w \Vdash \neg\psi$ iff $\mathcal{M}, w \nVdash \psi$ iff $\mathcal{M}_{[x \to w]} \nvDash ST_x(\psi)$ [by I.H.] iff $\mathcal{M}_{[x \to w]} \models \neg ST_x(\psi)$ iff $\mathcal{M}_{[x \to w]} \models ST_x(\neg\psi)$ iff $\mathcal{M}_{[x \to w]} \models ST_x(\varphi)$.

Case II: $\varphi := \psi \vee \chi$: Similar

Case III : $\varphi := \Diamond\psi$.

(a) Suppose $\mathcal{M}, w \Vdash \varphi$, that is $\mathcal{M}, w \Vdash \Diamond\psi$.
Then, there exists v in $\mathcal{M}$ s.t. wRv and $\mathcal{M}, v \Vdash \psi$. Now, consider two distinct variables, x and y, say.
Since wRv in $\mathcal{M}$, we have:
$\mathcal{M}_{[x \to w,\, y \to v]} \models Rxy$. Also, by I.H. we have $\mathcal{M}_{[y \to v]} \models ST_y(\psi)$. Then, we have $\mathcal{M}_{[x \to w,\, y \to v]} \models ST_y(\psi)$ as x does not occur free in $ST_y(\psi)$. Thus,
$\mathcal{M}_{[x \to w,\, y \to v]} \models Rxy \wedge ST_y(\psi)$.
Then, $\mathcal{M}_{[x \to w]} \models \exists y(Rxy \wedge ST_y(\psi))$,
that is, $\mathcal{M}_{[x \to w]} \models ST_x(\Diamond\psi)$,
that is, $\mathcal{M}_{[x \to w]} \models ST_x(\varphi)$.

(b) Conversely, suppose that $\mathcal{M}_{[x \to w]} \vDash ST_x(\varphi)$, that is $\mathcal{M}_{[x \to w]} \vDash ST_x(\Diamond \psi)$. Then,

$\mathcal{M}_{[x \to w]} \vDash \exists y (Rxy \wedge ST_y(\psi))$. Then,

$\mathcal{M}_{[x \to w, y \to v]} \vDash Rxy \wedge ST_y(\psi)$ for some v in $\mathcal{M}$. Then we have:

$$\mathcal{M}_{[x \to w, y \to v]} \vDash Rxy \quad \text{and} \quad \mathcal{M}_{[x \to w, y \to v]} \vDash ST_y(\psi)$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$w R v \quad \text{and} \quad \mathcal{M}_{[y \to v]} \vDash ST_y(\psi)$$

$$\Downarrow \text{ I.H.}$$

$$\mathcal{M}, v \Vdash \psi.$$

So, combining both, there is v in $\mathcal{M}$ s.t. wRv and $\mathcal{M}, v \Vdash \psi$. Thus $\mathcal{M}, w \Vdash \Diamond \psi$, that is, $\mathcal{M}, w \Vdash \varphi$. This completes the proof.

Applications.

- Compactness Theorem: An infinite set of modal formulas is satisfiable iff every finite subset of it is satisfiable. (H.W.)

- Lowenheim-Skolem Theorem: If a set of modal formulas is satisfiable in at least one infinite model, then it is satisfiable in models of every infinite cardinality. (H.W.)

Now, let us ask the converse question. Can every first-order formula (with appropriate parameters) be equivalent to a translation of a modal formula?

NO!

Modal formulas are invariant for bisimulation but first-order formulas may not be. Consider:

$$\varphi(x): \exists y_1 \exists y_2 \exists y_3 (y_1 \neq y_2 \wedge y_1 \neq y_3 \wedge y_2 \neq y_3 \wedge Rxy_1 \wedge Rxy_2 \wedge Ry_1y_3 \wedge Ry_2y_3)$$

This gives us a counter-example (See the model discussion earlier).

Q. What are those first-order formulas which are equivalent to the translation of modal formulas?

A. The first-order formulas that are invariant under bisimulation.

- Modal logic is the bisimulation-invariant fragment of First-order logic.

What is the definition of such formulas?

A first order formula $\varphi(x)$ is invariant under bisimulation if for all Kripke models $\mathcal{M}$ and $\mathcal{N}$ and for all states w in $\mathcal{M}$ and v in $\mathcal{N}$, and all bisimulations Z between $\mathcal{M}$ and $\mathcal{N}$ s.t. wZv, we have:

$$\mathcal{M}_{[x \mapsto w]} \models \varphi(x) \quad \text{iff} \quad \mathcal{N}_{[x \mapsto v]} \models \varphi(x).$$

The Theorem.

Let $\varphi(x)$ be a first-order formula. Then $\varphi(x)$ is invariant under bisimulation iff $\varphi(x)$ is logically equivalent to a standard translation of a modal formula.

- The proof is outside the scope of this course.

Correspondence Theory :

Modal logic Syntax :

$$\varphi, \psi ::= p \mid \bot \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \to \psi \mid \varphi \leftrightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \quad ; \quad p \in P .$$

Modal logic Semantics :

Models : $\mathcal{M} : (W, R, V)$

- W → a n.e. set of states
- R → $\subseteq W \times W$
- V → a map from P to 2^W.

Truth definition : $\mathcal{M}, w \models \varphi$

Focus on frames :

$F : (W, R)$

Q : Can we express properties of this relation R in terms of modal formulas ?

Satisfiability and Validity :

- A modal formula φ is <u>satisfiable</u> if there is a model $\mathcal{M}$ and a world w in $\mathcal{M}$ s.t. $\mathcal{M}, w \models \varphi$.

- A modal formula φ is <u>valid</u> if for every model $\mathcal{M}$ and for every world w in $\mathcal{M}$, $\mathcal{M}, w \models \varphi$.

Some variants of the notion of validity:

- Given a model $\mathcal{M} = (W, R, V)$, we call a formula φ <u>$\mathcal{M}$-valid</u> ($\mathcal{M} \models \varphi$) if for every $w \in W$, $\mathcal{M}, w \models \varphi$.

- Given a frame $F = (W, R)$, we call a formula φ <u>F-valid</u> ($F \models \varphi$) if for every model $\mathcal{M} = (F, V)$, φ is $\mathcal{M}$-valid.

- A modal formula φ is said to <u>characterize</u> a class of frames, $\mathcal{C}$, say, if $\mathcal{C} = \{F \mid \varphi \text{ is } F\text{-valid}\}$.

Examples.

1. Let $\mathcal{C} = \{(W, R) : R \text{ is reflexive}\}$

Q: Can we find a modal formula that characterizes $\mathcal{C}$?

A: Yes. $\Box p \to p$ (p is a propositional letter)

Proof: We need to show that for any frame F, $F \models \Box p \to p$ iff R_F is reflexive.

- Let F be a frame s.t. R_F is reflexive.

To show: $F \models \Box p \to p$.

So, we have to show that for all models M, based on F, and for all worlds w in M, $M, w \models \Box p \to p$.

Suppose $M, w \models \Box p$. To show: $M, w \models p$.

Since R_F is reflexive, $w R_F w$.

So, as $M, w \models \Box p$, $M, w \models p$ as well. We are done.

- Conversely, let F be a frame s.t. $F \models \Box p \to p$. To show that R_F is reflexive.

Let us prove a contrapositive statement, that is, if R_F is not reflexive then $F \not\models \Box p \to p$. Then we will be done.

Let F be a frame s.t. R_F is not reflexive. To show $F \not\models \Box p \to p$. It is enough to construct a model $\mathcal{M}$ based on the frame F and a world w in $\mathcal{M}$ s.t. $\mathcal{M}, w \not\models \Box p \to p$. Now, as R_F is not reflexive there is some $w \in W$ s.t. $w \not R_F w$. Consider the model $\mathcal{M} = (F, V)$, where $V(p) = W \setminus \{w\}$.

Then, $\mathcal{M}, w \models \Box p$, but $\mathcal{M}, w \not\models p$.

So, $\mathcal{M}, w \not\models \Box p \to p$.

So, $F \not\models \Box p \to p$.

This completes the proof.

2. Let $\mathcal{D} = \{(W, R) \mid R$ is transitive$\}$

Q. Can we characterize $\mathcal{D}$ by a modal formula?

A. Yes. $\Box p \to \Box\Box p$.

Proof. We need to show that for all frames F, $F \vDash \Box p \to \Box\Box p$ iff R_F is transitive.

(a) Let F be a frame s.t. R_F is transitive. To show $F \vDash \Box p \to \Box\Box p$. Let M be a model based on F and w be a world in M. To show $M, w \vDash \Box p \to \Box\Box p$

Suppose $M, w \vDash \Box p$. To show: $M, w \vDash \Box\Box p$.

Now, $M, w \vDash \Box\Box p$ if for all v with $w R_F v$, $M, v \vDash \Box p$, and $M, v \vDash \Box p$ if for all u with $v R_F u$, $M, u \vDash p$ ——— (*)

Now, since R_F is transitive, whenever

$w R_F v$ and $v R_F u$, we have $w R_F u$.
And then, as $\mathcal{M}, w \models \Box p$, we have:
$\mathcal{M}, u \models p$. To show $(*)$, take any
v s.t $w R_F v$ and u s.t. $v R_F u$.
We have, by our assumptions above,
$\mathcal{M}, u \models p$. So, we have $\mathcal{M}, w \models \Box\Box p$,
and, we are done.

(b) Conversely suppose that $F \models \Box p \to \Box\Box p$
To show that R_F is transitive.
We prove this contrapositively.
Suppose that R_F is not transitive.
To show: $F \not\models \Box p \to \Box\Box p$. It is
enough to show the existence of
a model $\mathcal{M}$ based on F and a
world w in $\mathcal{M}$, s.t. $\mathcal{M}, w \not\models \Box p \to \Box\Box p$.
Since R_F is not transitive, there

are $w, v, u \in W$ s.t. $w R_F v$ and $v R_F u$, but $w \not R_F u$.

Let $\mathcal{M}$ be a model based on F with V given by: $V(p) = W \setminus \{u\}$.

Then, $\mathcal{M}, w \models \Box p$ and $\mathcal{M}, w \not\models \Box\Box p$.

So, $\mathcal{M}, w \not\models \Box p \to \Box\Box p$. So $F \not\models \Box p \to \Box\Box p$.

This completes the proof.

H.W.

1. (a) R_F is serial

(b) R_F is symmetric.

2. What conditions on R_F do these formulas characterize. Justify your answer.

(a) $\Diamond p \to \Box p$

(b) $\Box(\Box p \to p)$

(c) $\Diamond\Box p \to \Box\Diamond p$