LECTURE 21

08.04.2024

Consequence relation

Let Γ be a set of modal formulas and φ be a modal formula.

Semantic consequence relation

$\Gamma \vDash \varphi$: For all Kripke models $\mathcal{M}$ and for all worlds w in $\mathcal{M}$, if $\mathcal{M}, w \vDash \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M}, w \vDash \varphi$.

Note: In the above definition, when Γ is empty, we say that φ is a valid formula (defined earlier) and we denote it by $\vDash \varphi$.

Deductive consequence relation

$\Gamma \vdash \varphi$ (Hilbert-style axiomatization):

If there is a sequence of formulas

$\varphi_1, \varphi_2, \ldots, \varphi_n$, such that $\varphi_n = \varphi$, and each φ_i is either an axiom, or a member of Γ or obtained by some rule.

Goal: To prove the following:

- if $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$ (Soundness Theorem)
- if $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$ (Completeness Theorem)

H.W. Prove the Soundness Theorem.

Completeness Theorem

Suppose $\Gamma \models \varphi$. To show that $\Gamma \vdash \varphi$.

Suppose not, that is $\Gamma \nvdash \varphi$.

[Introduce the concept of consistent sets of formulas as earlier: A set of formulas Λ is said to be inconsistent if there is a formula φ s.t. $\Lambda \vdash \varphi$ and $\Lambda \vdash \neg\varphi$.

Λ is consistent if it is not inconsistent.]

Now, since $\Gamma \nvdash \varphi$, then $\Gamma \cup \{\neg\varphi\}$ is consistent. Check!

Claim: Any consistent set of formulas has a model.

Using this claim, we have that $\Gamma \cup \{\neg\varphi\}$ has a model. This contradicts that $\Gamma \models \varphi$

Hence, we have that $\Gamma \vdash \varphi$. This completes the proof (modulo the claim above).

Let us now focus on the claim:

Any consistent set of formulas has a model

Proof of the claim:

Let Γ be a consistent set of formulas

1. Extend Γ to a consistent and complete (maximally consistent) set Δ, say.
[Lindenbaum Lemma] H.W.

2. Δ has a model [To do now]

First of all, let us note that the maximal consistent sets given above should have the natural properties in terms of the boolean connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ [look at the definition of model set : 1(a), (b), 2(a), (b), (c), (d)]. To get all these, we would need the axiom system for Classical Propositional Logic (CPL). To include this inside the axiom system for modal logic, we consider a notion of substitution

Substitution

A substitution is a map $\sigma : \mathcal{P} \rightarrow \mathcal{L}$ where $\mathcal{P}$ is the set of propositional letters and $\mathcal{L}$ is the set of modal formulas. Given σ, we have $\hat{\sigma} : \mathcal{L} \rightarrow \mathcal{L}$ as follows:

$$\hat{\sigma}(p) = \sigma(p)$$
$$\hat{\sigma}(\bot) = \bot$$
$$\hat{\sigma}(\neg\varphi) = \neg\hat{\sigma}(\varphi)$$

$$\hat{\sigma}(\varphi \vee \psi) = \hat{\sigma}(\varphi) \vee \hat{\sigma}(\psi)$$
$$\hat{\sigma}(\varphi \wedge \psi) = \hat{\sigma}(\varphi) \wedge \hat{\sigma}(\psi)$$
$$\hat{\sigma}(\varphi \rightarrow \psi) = \hat{\sigma}(\varphi) \rightarrow \hat{\sigma}(\psi)$$
$$\hat{\sigma}(\varphi \leftrightarrow \psi) = \hat{\sigma}(\varphi) \leftrightarrow \hat{\sigma}(\psi)$$

$\hat{\sigma}(\Diamond \varphi) = \Diamond \hat{\sigma}(\varphi)$ $\qquad$ $\hat{\sigma}(\Box \varphi) = \Box \hat{\sigma}(\varphi)$.

Note: σ can be uniquely extended to $\hat{\sigma}$, so we denote $\hat{\sigma}$ by σ.

Example

$\sigma(p) = \Box \Diamond (p \wedge q)$

$\sigma(q) = p \wedge \Box q$.

1. $\sigma(\Box (p \wedge q) \to \Box p)$

$= (\Box(\Box \Diamond (p \wedge q) \wedge (p \wedge \Box q)) \to \Box \Box \Diamond (p \wedge q)$

2. $\sigma(p \to (q \to p))$

$= \Box \Diamond (p \wedge q) \to ((p \wedge \Box q) \to \Box \Diamond (p \wedge q))$

Propositional tautology in modal logic.

A modal formula φ is a propositional tautology if $\varphi = \sigma(\alpha)$, where α is a propositional logic formula and a tautology in CPL and σ is a substitution.

We are now ready with the first set of axioms and rules that we need:

Axiom (1) All propositional tautologies

Rule (1) Modus Ponens.

We now focus on finding a model for an MCS $\Delta \supseteq \Gamma$ which would show that Γ has a model and the proof would be complete. Thus we have to find $\mathcal{M} = (W, R, V)$ and a world $w \in W$ such that the following holds:

$\mathcal{M}, w \models \varphi$ iff $\varphi \in w$ (Truth lemma)

Q. How do we get such an $\mathcal{M}$?

Let us define $\mathcal{M}$ as follows.

W is the set of all MCS's.

V is defined by: $V(p) = \{w \mid p \in w\}$.

R is defined by: wRu iff for all modal formulas φ, $\varphi \in u$ implies $\Diamond\varphi \in w$.

So, we have our required $\mathcal{M} = (W, R, V)$. And the required world is given by Δ, itself. Why?

Because, if we can show the truth lemma for $\mathcal{M}$: $\mathcal{M}, w \models \varphi$ iff $\varphi \in w$, then we would have $\mathcal{M}, \Delta \models \varphi$ for all $\varphi \in \Delta$, which in turn would give us $\mathcal{M}, \Delta \models \gamma$ for all $\gamma \in \Gamma$, and hence, the consistent set Γ that we started with, has a model. So, we prove the truth lemma now.

Proof of '$\mathcal{M}, w \models \varphi$ iff $\varphi \in w$' -

We prove this by induction on the size of of the formula φ.

Base Case: $\varphi := p$. It follows from the definition of V: $\mathcal{M}, w \models p$ iff $p \in w$.

I.H. : Suppose the result holds for all formulas φ with size $\leq n$.

I.S. : Let φ be a formula of size $n+1$.

Case 1 : $\varphi := \neg\psi$

$\mathcal{M}, w \models \varphi$ iff $\mathcal{M}, w \models \neg\psi$ iff $\mathcal{M}, w \not\models \psi$ iff $\psi \notin w$ (I.H.) iff $\neg\psi \in w$ (MCS) iff $\varphi \in w$.

Case 2 : $\varphi := \psi \vee \chi$

$\mathcal{M}, w \models \varphi$ iff $\mathcal{M}, w \models \psi \vee \chi$ iff $\mathcal{M}, w \models \psi$ or $\mathcal{M}, w \models \chi$ iff $\psi \in w$ or $\chi \in w$ (I.H.) iff $\psi \vee \chi \in w$ (MCS) iff $\varphi \in w$.

Case 3 : $\varphi := \Diamond\psi$.

We have to show that $\mathcal{M}, w \models \Diamond\psi$ iff $\Diamond\psi \in w$.

- Suppose $\mathcal{M}, w \models \Diamond\psi$. T.P. $\Diamond\psi \in w$.

$\mathcal{M}, w \models \Diamond\psi$

implies there is v in $\mathcal{M}$ s.t. $w R v$
and $\mathcal{M}, v \models \psi$

implies there is v in $\mathcal{M}$ s.t. $w R v$
and $\psi \in v$ (I.H.).

implies $\Diamond \psi \in w$ (by definition of R).

implies $\varphi \in w$

- Conversely, suppose $\varphi \in w$, that is, $\Diamond \psi \in w$.
T.P. $\mathcal{M}, w \models \Diamond \psi$; that is, there is a v in $\mathcal{M}$ s.t. $w R v$ and $\mathcal{M}, v \models \psi$.
By I.H., it is enough to show that there is a v s.t. $w R v$ and $\psi \in v$.
Now, $w R v$ means that for all modal formulas φ, $\varphi \in v$ implies $\Diamond \varphi \in w$.

Thus, our assumption is $\Diamond \psi \in w$
And, we need to show the existence of a v s.t. $\psi \in v$ and for all modal formulas φ, $\varphi \in v$ implies $\Diamond \varphi \in w$.

Let us first prove the following:

Observation: wRv iff for all modal formulas φ, $\Box\varphi \in w$ implies $\varphi \in v$.

Proof of the observation.

- Let wRv. Let φ be any modal formula s.t. $\Box\varphi \in w$. To show $\varphi \in v$. Suppose not. So, $\varphi \notin v$. Then $\neg\varphi \in v$. So, $\Diamond\neg\varphi \in w$.

[Axiom (2): $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$]

Then, $\neg\Box\varphi \in w$ (Check!).

This is a contradiction to the consistency of w. So, we have our result.

- Conversely, suppose that $\Box\varphi \in w$ implies $\varphi \in v$ for all modal formulas φ. To show that wRv, that is $\varphi \in v$ implies $\Diamond\varphi \in w$ for all modal formulas φ. Let φ be a modal formula s.t. $\varphi \in v$. To show, $\Diamond\varphi \in w$. Suppose not. Then $\Diamond\varphi \notin w$. Then, $\neg\Diamond\varphi \in w$. Then, by Axiom (2), $\Box\neg\varphi \in w$. (Check!). So, we have by the given condition, $\neg\varphi \in v$. This contradicts the consistency of v. Hence we have our result.

This completes the proof of the observation.

In the next class, we will get back to the main proof.