Lecture 22

A decision procedure for the satisfiability problem for modal logic.

Problem: Given φ_0, check whether it is satisfiable.

Method: A tableau-based decision procedure.

φ_0-tableau: It is a tree $T = (V, E, v_0, \lambda)$, where $\lambda : V \to AT_{\varphi_0}$, satisfying certain conditions.

What is AT_{φ_0}?

Given φ_0, we have $SF(\varphi_0)$, the set of all subformulas of φ_0.

A set $A \subseteq SF(\varphi_0)$ is said to be down-closed if the following conditions hold:

- if $\varphi_1 \vee \varphi_2 \in A$, then $\varphi_1 \in A$ or, $\varphi_2 \in A$.
- if $\varphi_1 \wedge \varphi_2 \in A$, then $\{\varphi_1, \varphi_2\} \subseteq A$.
- if $\neg p \in A$, then $p \notin A$.

$AT(\varphi_0)$ = the set of all down-closed subsets of $SF(\varphi_0)$.

Given any $A \subseteq SF(\varphi_0)$, define the down-closure of A ($dc(A)$) as follows:

$$dc(A) = \{B \subseteq SF(\varphi_0) \mid A \subseteq B \in AT(\varphi_0)\}$$

Assumption: φ_0 is in negation-normal form (NNF).

What is NNF?

A modal formula φ is said to be in NNF is it has the following syntax: $p \mid \neg p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \Box \varphi \mid \Diamond \varphi$.

Lemma: Any modal formula is equivalent to a modal formula in NNF

H.W. Prove this lemma.

[Hint: $\neg \Box \varphi := \Diamond \neg \varphi$; $\neg \Diamond \varphi := \Box \neg \varphi$.
$\neg(\varphi \vee \psi) := \neg \varphi \wedge \neg \psi$; $\neg(\varphi \wedge \psi) := \neg \varphi \vee \neg \psi$]

Let us now get back to our φ_0-tableau.
It is a tree (V, E, v_0, λ), where v_0 is the root and $\lambda : V \to AT(\varphi_0)$ satisfies:

* $\varphi_0 \in \lambda(v_0)$.
* whenever $v E v'$, if $\Box \varphi \in \lambda(v)$, then $\varphi \in \lambda(v')$
* whenever $\Diamond \varphi \in \lambda(v)$, there is a v' such that $v E v'$ and $\varphi \in \lambda(v')$.

Algorithm:

Step 1: if $dc(\{\varphi_0\}) = \Phi$, STOP FAIL else pick $A_0 \in dc(\{\varphi_0\})$ and construct $T = (\{v_0\}, \emptyset, v_0, \lambda(v_0) = A_0)$.

Step 2 : while [there exist $v \in V$, with $\Diamond \varphi \in \lambda(v)$ such that there is no v' with $(v, v') \in E$ and $\varphi \in \lambda(v')$]

do

consider $B = \{\psi \mid \Box \psi \in \lambda(v)\} \cup \{\varphi\}$

if $dc(B) = \emptyset$ STOP FAIL

else pick $B_0 \in dc(B)$.

define $T = (V', E', v_0, \lambda')$,

$$V' = V \cup \{v'\}$$

$$E' = E \cup \{(v, v')\}$$

$$\lambda'(\bar{v}) = \begin{cases} \lambda(\bar{v}) & \text{if } \bar{v} \in V \\ B_0, & \text{otherwise.} \end{cases}$$

Observation : The algorithm terminates.

To show this, let us define the notion

of modal depth of a formula φ, denoted by $md(\varphi)$: $md(p) = md(\neg p) = 0$;

$md(\varphi \wedge \psi) = md(\varphi \vee \psi) = \max\{md(\varphi), md(\psi)\}$;

$md(\Box \varphi) = md(\Diamond \varphi) = md(\varphi) + 1$.

Now, let T denote the tableau formed. We have: (i) $\mathrm{depth}(T) \leq md(\varphi_0)$, where the problem is to check the satisfiability of φ_0, (ii) the branching degree of T is linear in the size of the input formula φ_0 (that is, the number of connectives and modal operators present in φ_0).

In other words, branching degree of T is $\leq O(|\varphi_0|)$.

Hence, the algorithm terminates.

Correctness of the algorithm.

Result: φ_0 is satisfiable iff the algorithm applied on φ_0 terminates successfully.

Proof. Suppose φ_0 is satisfiable. Let $\mathcal{M} = (W, R, V)$ and $w_0 \in W$ be such that $\mathcal{M}, w_0 \models \varphi_0$. We can assume that $\mathcal{M}$ is a tree with root w_0 (by a previous proposition). We now have to show that the algorithm terminates successfully. To this end, let us first define a function $\mu : W \to 2^{SF(\varphi_0)}$ as follows: $\mu(w) = \{\varphi \in SF(\varphi_0) \mid \mathcal{M}, w \models \varphi\}$.

Then $\mu(w) \in AT_{\varphi_0}$ (Check!)

Also, $\varphi_0 \in \mu(w_0) \in dc(\{\varphi_0\})$. So, $dc(\{\Phi_0\}) \neq \emptyset$. So, Step 1 is taken care of. Now we have to take care of Step 2. This we will continue in the next class.