LECTURE 23

15.04.2024

Some definitions

- Size of a Kripke model $\mathcal{M} = (W, R, V)$ is the cardinality of W, denoted by $|W|$.

- Given a modal formula φ, the set of subformulas of φ is defined as follows:

$$Sub(p) = \{p\},$$

$$Sub(*\varphi) = \{*\varphi\} \cup Sub\{\varphi\}, \quad * \in \{\neg, \Diamond, \Box\}$$

$$Sub(\varphi \bullet \psi) = \{\varphi \bullet \psi\} \cup Sub\{\varphi\} \cup Sub(\psi), \quad \bullet \in \{\wedge, \vee, \to\}$$

Proof of strong finite model property:

Let φ be a modal formula. Let $Sub(\varphi)$ denote the set of all subformulas of φ. Then $Sub(\varphi)$ is a finite set of formulas. We assume that φ is satisfiable. Then, there is a model $\mathcal{M} = (W, R, V)$ and a world w in $\mathcal{M}$ s.t. $\mathcal{M}, w \models \varphi$. Now, the question is how to make the size of $\mathcal{M}$, small, in fact, make it finite. A natural way is to think about partitioning the set W. Now, to form a partition, we need to define an equivalence relation on W.

And, we also need to relate the formula φ in some way so that the satisfaction of the formula φ gets preserved in the smaller model.

Let us define an equivalence relation on W with respect to the set $Sub(\varphi)$ as follows:

Let $u, v \in W$. We say u is equivalent to v with respect to $Sub(\varphi)$ if the following holds:

for all $\psi \in Sub(\varphi)$, $\mathcal{M}, u \models \psi$ iff $\mathcal{M}, v \models \psi$.

We denote this relation by $u \sim_\varphi v$.

Claim: $\sim_\varphi$ is an equivalence relation.

H.W. Prove the claim.

Now, $\sim_\varphi$ partitions W into equivalence classes. Let $[v]$ denote the equivalence class of v in W, and let $W_\sim =$

$\{[v] \mid v \in W\}$.

Q. How many elements does $W_\sim$ have?

Let us define a map $f : W_\sim \to 2^{Sub(\varphi)}$ as follows: $f([v]) = \{\psi \in Sub(\varphi) : M, v \models \psi\}$

So, $f([v]) \subseteq Sub(\varphi)$.

(i) f is well-defined.

To show that, if $u \sim v$, then $f([u]) = f([v])$

Let $u, v \in W$ s.t $u \sim v$. Then, $M, u \models \psi$ iff $M, v \models \psi$ for all $\psi \in Sub(\varphi)$. So, by definition of f, $f([u]) = f([v])$.

(ii) f is injective.

Let $u, v \in W$ s.t. $f([u]) = f([v])$

Then, $\{\psi \in Sub(\varphi) : \mathcal{M}, u \models \psi\}$

$= \{\psi \in Sub(\varphi) : \mathcal{M}, v \models \psi\}$.

So, $\mathcal{M}, u \models \psi$ iff $\mathcal{M}, v \models \psi$ for all $\psi \in Sub(\varphi)$.

So, $u \sim v$, that is, $[u] = [v]$.

Thus f is an injective map from $W_\sim$ to $2^{Sub(\varphi)}$. Now $|2^{Sub(\varphi)}| = 2^{|\varphi|}$.

Hence, $|W_\sim| \leq 2^{|\varphi|}$.

Thus starting from W, we get to a finite bounded set $W_\sim$. Now, we need to define a binary relation $R_\sim$ on $W_\sim$ and a valuation function on $V_\sim$, say, such that the satisfaction of formulas in $Sub(\varphi)$ do not get affected, that is, $\mathcal{M}, v \models \psi$ iff $\mathcal{M}_\sim, [v] \models \psi$ for all $\psi \in Sub(\varphi)$, where

$M_\sim = (W_\sim, R_\sim, V_\sim)$.

- Let us first define $V_\sim$ as follows:

$V_\sim(p) = \{[v] : v \in V(p)\}$, $p \in \mathcal{P}$.

Q. How do we define $R_\sim$?

Let us postpone this discussion for now and get to the proof of the following:

Lemma: For all formulas $\psi \in Sub(\varphi)$ and for all v in M, $M, v \models \psi$ iff $M_\sim, [v] \models \psi$.

Proof: We prove by applying induction on the size of ψ.

Base Case: $\psi = p$. Then, $M, v \models p$ iff $v \in V(p)$ iff $[v] \in V_\sim(p)$ (by definition of $V_\sim$) iff $M_\sim, [v] \models p$.

Induction Hypothesis: Suppose the result holds for all formulas ψ of size $\leq n$.

Induction Step: Let φ be a formula of size $m+1$.

Case 1: $\varphi = \neg\chi$. Then, $\mathcal{M}, v \models \neg\chi$ iff $\mathcal{M}, v \not\models \chi$ iff $\mathcal{M}_\sim, [v] \not\models \chi$ (I.H.) iff $\mathcal{M}_\sim, [v] \models \neg\chi$.

Case 2: $\psi = \eta \vee \delta$. Then, $\mathcal{M}, v \models \eta \vee \delta$
iff $\mathcal{M}, v \models \eta$ or $\mathcal{M}, v \models \delta$
iff $\mathcal{M}_\sim, [v] \models \eta$ or $\mathcal{M}_\sim, [v] \models \delta$ (I.H.)
iff $\mathcal{M}_\sim, [v] \models \eta \vee \delta$.

Case 3: $\psi = \Diamond\chi$.

- Suppose $\mathcal{M}, v \models \Diamond\chi$. Then there exists u in $\mathcal{M}$ s.t vRu and $\mathcal{M}, u \models \chi$.
To show that $\mathcal{M}_\sim, [v] \models \Diamond\chi$, that is to show that there exists $[z]$ in $\mathcal{M}_\sim$ s.t. $[v] R_\sim [z]$, and $\mathcal{M}_\sim, [z] \models \chi$.

Now, since $\mathcal{M}, u \models \chi$, by I.H., $\mathcal{M}_\sim, [u] \models \chi$.
So, if we can show that $[v] R_\sim [u]$,

we are done.

Condition (1) on $R_\sim$:

if $v R u$ then $[v] R_\sim [u]$.

Let us assume (1). Then, we have our required result, that is, $\mathcal{M}_\sim, [v] \models \Diamond \chi$.

- Conversely, suppose that $\mathcal{M}_\sim, [v] \models \Diamond \chi$.

To show, $\mathcal{M}, v \models \Diamond \chi$. Now, since $\mathcal{M}_\sim, [v] \models \Diamond \chi$, there exists $[u]$ in $\mathcal{M}_\sim$, s.t. $[v] R_\sim [u]$ and $\mathcal{M}_\sim, [u] \models \chi$.

By I.H., we have that $\mathcal{M}, u \models \chi$.

We somehow need to show that $\mathcal{M}, v \models \Diamond \chi$.

Condition (2) on $R_\sim$:

If $[v] R_\sim [u]$, then for all $\Diamond \delta \in Sub(\varphi)$, if $\mathcal{M}, u \models \delta$, then $\mathcal{M}, v \models \Diamond \delta$.

Let us assume Condition (2). Then, we have: $\mathcal{M}, v \models \Diamond \chi$.

This completes the proof once we have an $R_\sim$ on $W_\sim$ satisfying conditions (1) and (2).

A definition of $R_\sim$ satisfying conditions (1) and (2):

- $[v] R_\sim [u]$ iff there exists $v' \in [v]$ and $u' \in [u]$, s.t. $v' R u'$.

H.W. Show that $R_\sim$ satisfies (1) and (2).

This completes the proof of strong finite model property.

Q. How do we get decidability from strong finite model property?

We start with a formula φ.
We have the bound $2^{|\varphi|}$. We consider all possible models of size $1, 2, 3, \ldots, 2^{|\varphi|}$ and check

whether φ is satisfiable in any such model. How do we check? We construct a Turing machine to generate all such models of size at most $2^{|\varphi|}$ and checking the satisfiability. If we get a satisfiable model we can say that 'φ is satisfiable'. If there are no models of φ till the size $2^{|\varphi|}$, we can say that 'φ is unsatisfiable' by the strong finite model property.

Thus, basic modal logic is decidable.

Note: First-order logic is undecidable.