Lecture 24

Till now we were discussing modal logic from the perspective of models of the form $\mathcal{M}: (W, R, V)$. Today we will concentrate on the notion of frames $F: (W, R)$.

Correspondence Theory

PML Syntax:

$$\varphi, \psi := p \mid \bot \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \varphi \leftrightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \quad , \quad p \in \mathcal{P}.$$

PML Semantics:

Models : $\mathcal{M} : (W, R, V)$

- W: n.e. set
- $R \subseteq W \times W$
- $V : \mathcal{P} \rightarrow 2^W$

Truth definition: $\mathcal{M}, w \models \varphi$.

We focus on frames $F: (W, R)$ and ask:

Can we express properties of this relation R in terms of modal formulas?

Satisfiability and validity:

* A modal formula φ is satisfiable if there is a model $\mathcal{M}$ and a world w in $\mathcal{M}$ such that $\mathcal{M}, w \models \varphi$.

* A modal formula φ is valid if for every model $\mathcal{M}$ and every world w in $\mathcal{M}$, $\mathcal{M}, w \models \varphi$.

Some variants of the notion of validity.

- Given a model $\mathcal{M} = (W, R, V)$, we call a formula φ $\mathcal{M}$-valid ($\mathcal{M} \models \varphi$) if for all $w \in W$, $\mathcal{M}, w \models \varphi$.
- Given a frame $F = (W, R)$, we call a formula φ F-valid ($F \models \varphi$) if for every model $\mathcal{M} = (F, V)$, φ is $\mathcal{M}$-valid.
- A modal formula φ is said to characterize a class of frames, $\mathcal{C}$, say, if $\mathcal{C} = \{F \mid \varphi \text{ is } F\text{-valid}\}$

Examples

1. Consider $\mathcal{C} = \{(W, R) : R \text{ is reflexive}\}$

Can we find a modal formula that characterizes C?

Yes, we can : $\Box p \to p$ (equivalently, $p \to \Diamond p$), where p is some propositional variable.

Claim : $\Box p \to p$ characterizes reflexive frames.

Proof : We need to show that for any frame F, $F \models \Box p \to p$ iff R_F is reflexive.

✱ Let F be a frame such that R_F is reflexive. To show that : $F \models \Box p \to p$. Then, we need to show that for all models $\mathcal{M}$, based on F, and for all worlds w in $\mathcal{M}$, $\mathcal{M}, w \models \Box p \to p$. Take some $\mathcal{M}$ and some w in $\mathcal{M}$. To show that $\mathcal{M}, w \models \Box p \to p$, we assume that $\mathcal{M}, w \models \Box p$. We have to show that $\mathcal{M}, w \models p$. Since, R_F is reflexive, $w R_F w$. So, as $\mathcal{M}, w \models \Box p$, $\mathcal{M}, w \models p$, and we are done.

✱ Conversely, suppose that F is a frame such that $F \models \Box p \to p$. To show that R_F is reflexive. We prove this contrapositively. Suppose R_F

is not reflexive. To show $F \not\Vdash \Box p \to p$. It is enough to construct a model $\mathcal{M}$ based on the frame F and a world w in $\mathcal{M}$ such that $\mathcal{M}, w \not\Vdash \Box p \to p$. Now, as R_F is not reflexive, there is some $w \in W$, such that $w \not R_F w$. Consider the model $\mathcal{M} = (F, V)$, where $V(p) = W \setminus \{w\}$. Then, $\mathcal{M}, w \Vdash \Box p$, but, $\mathcal{M}, w \not\Vdash p$. So, $\mathcal{M}, w \not\Vdash \Box p \to p$. So, $F \not\Vdash \Box p \to p$. This completes the proof.

2. Consider $\mathcal{D} = \{(W, R) \mid R \text{ is transitive}\}$.

Can we find a modal formula that characterizes $\mathcal{D}$?

Yes, we can. $\Box p \to \Box\Box p$ (equivalently, $\Diamond\Diamond p \to \Diamond p$) for some propositional variable p.

Claim: $\Box p \to \Box\Box p$ characterizes transitive frames.

Proof: We need to show that for any frame F,

$F \models \Box p \to \Box\Box p$ iff R_F is transitive.

* Suppose F is a frame such that R_F is transitive. To show that $F \models \Box p \to \Box\Box p$. Let $\mathcal{M}$ be a model based on F, and w be a world in $\mathcal{M}$. To show $\mathcal{M}, w \models \Box p \to \Box\Box p$. Let $\mathcal{M}, w \models \Box p$. We have to show $\mathcal{M}, w \models \Box\Box p$.

Now, $\mathcal{M}, w \models \Box\Box p$ if for all v with $w R_F v$, $\mathcal{M}, v \models \Box p$, and $\mathcal{M}, w \models \Box p$ if for all u with $v R_F u$, $\mathcal{M}, u \models p$. —— (*)

Since R_F is transitive, whenever $w R_F v$ and $v R_F u$, we have $w R_F u$. And, then as $\mathcal{M}, w \models \Box p$, we have $\mathcal{M}, u \models p$. To show (*), take any v such that $w R_F v$ and u such that $v R_F u$. We have, by our assumption above, $\mathcal{M}, u \models p$. So, we have, $\mathcal{M}, w \models \Box\Box p$; and we are done.

* Conversely, suppose that $F \models \Box p \to \Box\Box p$. To show that R_F is transitive. We prove this contrapositively. Suppose that R_F is not transitive. To show: $F \not\models \Box p \to \Box\Box p$. It is enough to construct a model $\mathcal{M} = (F, V)$ and a world w in

$\mathcal{M}$, such that $\mathcal{M}, w \not\models \Box p \to \Box\Box p$. Since R_F is not transitive, there are $w, v, u \in W$ such that $w R_F v$ and $v R_F u$, but $w \not R_F u$. Now consider a valuation V such that $V(p) = W \setminus \{u\}$. Then, we have $\mathcal{M}, w \models \Box p$ and $\mathcal{M}, w \not\models \Box\Box p$. So, $\mathcal{M}, w \not\models \Box p \to \Box\Box p$, hence, $F \not\models \Box p \to \Box\Box p$.

This completes the proof.

H.W.

1. (a) Characterize the class of all symmetric frames.

   (b) Characterize the class of all serial frames.

2. What conditions on R_F do the following formulas characterize? Justify your answer.

   (a) $\Diamond p \to \Box p$

   (b) $\Diamond p \to \Box \Diamond p$

   (c) $\Box\Box p \to \Box p$

   (d) $\Box(\Box p \to p)$

   (e) $\Diamond \Box p \to \Box \Diamond p$.

Note: Here and above, p can be replaced by any modal formula φ.