Lecture 25

Consequence relation

Let Γ be a set of modal formulas and φ be a modal formula.

Semantic consequence relation

$\Gamma \vDash \varphi$: For all Kripke models M and for all worlds w in M, if $M, w \vDash \gamma$ for all $\gamma \in \Gamma$, then $M, w \vDash \varphi$.

Note : When Γ is empty, we say φ is a validity (defined earlier) and we denote this by $\vDash \varphi$.

Deductive consequence relation

$\Gamma \vdash \varphi$ (Hilbert-style axiomatisation) : If there is a sequence of formulas $\varphi_1, \varphi_2, \dots, \varphi_n$, such that $\varphi_n = \varphi$, and each φ_i is either an axiom, or a member of Γ or obtained by some rule.

Goal: To prove the following.

Soundness theorem: If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$

Completeness theorem: If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

H.W. Prove the soundness theorem.

Completeness theorem:

Suppose $\Gamma \models \varphi$. To show that $\Gamma \vdash \varphi$.

Suppose not, that is $\Gamma \nvdash \varphi$. [Introduce the concept of consistent sets of formulas as earlier: A set of formulas Λ is said to be inconsistent if there is a formula φ such that $\Lambda \vdash \varphi$ and $\Lambda \vdash \neg\varphi$.

Λ is consistent if it is not inconsistent.]

Now, since $\Gamma \nvdash \varphi$, then $\Gamma \cup \{\neg\varphi\}$ is consistent. (Check!)

Claim: Any consistent set of formulas has a model

Using this claim, we have that $\Gamma \cup \{\neg \varphi\}$ has a model. This contradicts the fact that $\Gamma \models \varphi$. So we have: $\Gamma \vdash \varphi$.

This completes the proof (modulo the claim above).

Let us now focus on the claim:

Let Γ be a consistent set of formulas.

We do the following now:

☆ 1. Extend Γ to a consistent and complete (maximally consistent) set Δ, say.
[Lindenbaum's Lemma] H. W.

Let us note that the maximal consistent sets of formulas given above should have the natural properties in terms of the Boolean connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ [Look at the definition of the model set: 1(a), (b); 2(a), (b), (c), (d).] To get all these we would need the axiom system for CPL. To include them inside the axiom system for modal logic, we consider a notion of substitution.

Substitution

A substitution is a map $\sigma : \mathcal{P} \to \mathcal{L}$, where $\mathcal{P}$ is the set of propositional variables and $\mathcal{L}$ is the set of modal formulas.

Given σ, we have $\hat{\sigma} : \mathcal{L} \to \mathcal{L}$ as follows:

$\hat{\sigma}(p) = \sigma(p)$; $\hat{\sigma}(\bot) = \bot$; $\hat{\sigma}(\neg\varphi) = \neg\hat{\sigma}(\varphi)$;

$\hat{\sigma}(\varphi \vee \psi) = \hat{\sigma}(\varphi) \vee \hat{\sigma}(\psi)$; $\hat{\sigma}(\varphi \wedge \psi) = \hat{\sigma}(\varphi) \wedge \hat{\sigma}(\psi)$

$\hat{\sigma}(\varphi \to \psi) = \hat{\sigma}(\varphi) \to \hat{\sigma}(\psi)$; $\hat{\sigma}(\varphi \leftrightarrow \psi) = \hat{\sigma}(\varphi) \leftrightarrow \hat{\sigma}(\psi)$

$\hat{\sigma}(\Box\varphi) = \Box\hat{\sigma}(\varphi)$; $\hat{\sigma}(\Diamond\varphi) = \Diamond\hat{\sigma}(\varphi)$.

Note: σ can be extended uniquely to $\hat{\sigma}$, so we denote $\hat{\sigma}$ by σ.

Examples

$\sigma(p) = \Box\Diamond(p \wedge q)$

$\sigma(q) = p \wedge \Box q$

1. $\sigma(\Box(p \wedge q) \to \Box p)$

$= (\Box(\Box\Diamond(p \wedge q) \wedge (p \wedge \Box q)) \to \Box\Box\Diamond(p \wedge q))$

2. $\sigma(p \to (q \to p))$

$= \Box\Diamond(p \wedge q) \to ((p \wedge \Box q) \to \Box\Diamond(p \wedge q))$

Propositional tautology in modal logic

A modal formula φ is a propositional tautology if $\varphi = \sigma(\alpha)$, where α is a propositional logic formula and a tautology in CPL and σ is a substitution function.

We are now ready with the first set of axioms and rules that we need.

Axioms

(1) All propositional tautologies

Rules

(1) Modus Ponens

★2. Show that Δ has a model.

We now focus on finding a model for the MCS $\Delta \supseteq \Gamma$ which would show us that Γ has a model and the proof would be complete. Thus we have to find $\mathcal{M} = (W, R, V)$ and a world $w \in W$ such that the following holds;

$$\mathcal{M}, w \models \varphi \text{ iff } \varphi \in \Delta \quad \text{(Truth lemma)}$$

How do we get such an $\mathcal{M}$?

Let us define $\mathcal{M}$ as follows:

W is the set of all MCS's.

V is defined by: $V(p) = \{ w \in W \mid p \in w \}$

R is defined by: $w R v$ iff for all modal formulas φ, $\varphi \in v$ implies $\Diamond \varphi \in w$.

So, we have $\mathcal{M} = (W, R, V)$. And the required world is given by Δ itself. (Why?)