Lecture 26

Let us first note that this model we constructed above is known as the canonical model. We are now ready to prove the following.

$\mathcal{M}, w \models \varphi$ iff $\varphi \in w$, where $\mathcal{M}$ is the canonical model. (Truth lemma).

Once we have the truth lemma, we would have : $\mathcal{M}, \Delta \models \varphi$ iff $\varphi \in \Delta$, which gives us a pointed model $(\mathcal{M}, \Delta)$ for Δ, and hence a pointed model $(\mathcal{M}, \Delta)$ for Γ as $\Delta \supseteq \Gamma$.

Proof of the truth lemma

We prove this by applying induction on the size of φ.

Base case: $\varphi := p$. The result follows from the definition of V. $\mathcal{M}, w \models p$ iff $w \in V(p)$ iff $p \in w$.

I.H.: Suppose the result holds for all formulas φ of size $\leq m$.

I.S.: Suppose φ is a formula of size $m+1$. Then we have the following cases

$\varphi := \neg\psi$. $\mathcal{M}, w \models \neg\psi$ iff $\mathcal{M}, w \not\models \psi$ iff $\psi \notin w$ (I.H) iff $\neg\psi \in w$ (MCS). So, $\mathcal{M}, w \models \varphi$ iff $\varphi \in w$.

$\varphi := \psi \vee \chi$ $\mathcal{M}, w \models \psi \vee \chi$ iff $\mathcal{M}, w \models \psi$ or, $\mathcal{M}, w \models \chi$, iff $\psi \in w$ or, $\chi \in w$ (I.H.) iff $\psi \vee \chi \in w$ (MCS). So, $\mathcal{M}, w \models \varphi$ iff $\varphi \in w$.

$\varphi : \Diamond\psi$. We have to show that $\mathcal{M}, w \models \Diamond\psi$ iff $\Diamond\psi \in w$.

- Suppose $\mathcal{M}, w \models \Diamond\psi$. To show $\Diamond\psi \in w$. Since $\mathcal{M}, w \models \Diamond\psi$, there is v in $\mathcal{M}$ with wRv and $\mathcal{M}, v \models \psi$. So, by I.H., there is v in $\mathcal{M}$ with wRv and $\psi \in v$. Then, by definition of R, $\Diamond\psi \in w$.

- Conversely, suppose that $\Diamond\psi \in w$. To show that $\mathcal{M}, w \models \Diamond\psi$. We have to show that there is v in $\mathcal{M}$ such that wRv and $\mathcal{M}, v \models \psi$ (by I.H., $\mathcal{M}, v \models \psi$ iff $\psi \in v$). So, we have to find a v in $\mathcal{M}$ such that wRv and $\psi \in v$. Thus our assumption is $\Diamond\psi \in w$. And, we need to show the existence of a v such that $\psi \in v$ and for all modal formulas φ, $\varphi \in v$ implies $\Diamond\varphi \in w$. Let us first prove the following

Observation: wRv iff for all modal formulas φ, $\Box\varphi \in w$ implies $\varphi \in v$.

Proof: $\Rightarrow$ Suppose wRv. Let φ be a modal formula such that $\Box\varphi \in w$. To show $\varphi \in v$. Suppose not. Then, $\neg\varphi \in v$. So $\Diamond\neg\varphi \in w$.

[Axiom 2: $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$]

Then $\neg\Box\varphi \in w$ (Check!)

This is a contradiction to the consistency of w. So we have our result, that is, $\varphi \in v$.

Thus for all modal formulas φ, whenever $\Box\varphi \in w$, $\varphi \in v$.

* Conversely, suppose that for all modal formulas φ, $\Box\varphi \in w$ implies $\varphi \in v$. To show that wRv, that is, for all modal formulas φ, $\varphi \in v$ implies $\Diamond\varphi \in w$. Let $\varphi \in v$. To show $\Diamond\varphi \in w$. Suppose not. Then, $\neg\Diamond\varphi \in w$. Then, by Axiom (2), $\Box\neg\varphi \in w$ (Check!). Then, $\neg\varphi \in v$, a contradiction. Hence, the result.

This completes the proof of the observation.

Let us now go back to the original proof. We have: $\Diamond\psi \in w$. We need to show the existence of an MCS v such that wRv and $\psi \in v$.

Let $v' = \{\psi\} \cup \{\chi : \Box\chi \in w\}$.

It is enough to show that v' is

consistent. Because then, we can take v to be an MCS extending v'. Such a v would satisfy both the conditions $\psi \in v$ and $w R v$.

Proof of v' being consistent

Suppose not. Then there exists a finite subset of v' that is inconsistent. So, there are $\chi_1, \chi_2, \cdots, \chi_n$, such that

$$\{\chi_1, \chi_2, \cdots, \chi_n\} \vdash \neg\psi \quad \text{(Check!)}$$

[H.W. If $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma \vdash \neg\varphi$]

Then, $\vdash (\chi_1 \wedge \chi_2 \wedge \cdots \wedge \chi_n) \to \neg\psi$.

[Rule (2): $\dfrac{\vdash \varphi}{\vdash \Box\varphi}$ (Generalisation)]

Then, $\vdash \Box((\chi_1 \wedge \chi_2 \wedge \cdots \wedge \chi_n) \to \neg\psi)$

[Axiom (3): $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$]

Then, $\vdash \Box(\chi_1 \wedge \chi_2 \wedge \cdots \wedge \chi_n) \to \Box\neg\psi$.

[Theorem: $\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$ (H.W.)]

Then, $\vdash (\Box\chi_1 \wedge \Box\chi_2 \wedge \cdots \Box\chi_n) \to \Box\neg\psi$.

Now, $\Box\chi_1 \wedge \Box\chi_2 \wedge \cdots \wedge \Box\chi_n \in w$.

So, $\Box\neg\psi \in w$. Then $\neg\Diamond\psi \in w$.

(by Axiom (2)). But $\Diamond\psi \in w$, so we have a contradiction. Hence, the result.

This completes the proof of truth lemma and hence, the completeness.

Axiom system for basic modal logic (PML)

Axiom 1 : All propositional tautologies.

Axiom 2 : $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$.

Axiom 3 : $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.

Rule 1 : $\dfrac{\varphi \quad \varphi\to\psi}{\psi}$ (M.P.)

Rule 2 : $\dfrac{\vdash \varphi}{\vdash \Box\varphi}$ (Gen.)

Can we get such sound and complete axiom systems for certain special class of models?

- Models based on reflexive frames

- Models based on transitive frames
- Models based on symmetric frames
- Models based on equivalence frames

In the proof above, we showed that $\Gamma \vdash \varphi$ iff $\Gamma \vDash \varphi$. For defining $\vDash$ we considered all Kripke models. Now we can consider some restricted class of models like above.

Examples

1. Reflexive models:
What would be the corresponding axiom system?

The proof idea follows from the completeness proof that we finished just now, but for this case we need to ensure that the canonical model is reflexive, that is R should have the property: wRw for all $w \in W$, the set of all MCS's. This means that for all modal formulas φ, $\varphi \in w$ implies $\Diamond\varphi \in w$ (equivalently,

$\Box\varphi \in w$ implies $\varphi \in w$).

Axiom T : $\Box\varphi \to \varphi$.

Thus, the axiom system for PML

+

Axiom T

will give us a complete axiomatization for the models based on reflexive frames.

2. Transitive models.

Axiom 4 : $\Box\varphi \to \Box\Box\varphi$

Then, the axiom system for PML

+

Axiom 4

will give us a complete axiomatization for the models based on transitive frames.

Some other examples.

1. System K (all models)
2. System T (reflexive models)
3. System S4 (reflexive and transitive models)
4. System S5 (models with equivalence relations)