Lecture 27

Modal logic of transitive closure

Basic modal logic talks about transition systems (Kripke models), the relation giving you the transitions from one point to another. In addition to expressing successor states we are also interested in reachability, that is, states reachable from the current state.

Can't we express reachability in PML?

No. We cannot even express reachability (path-connectedness) in FOL, and PML is only a fragment of the 2-variable FOL. Thus we need to go beyond basic modal logic to talk about reachability.

How can we express reachability in terms of the transition relation (of the Kripke model)?

Consider the reflexive transitive closure of the relation in the Kripke model. That expresses reachability and let us introduce modal logic of transitive closure which allows us to express transitive closure of a relation.

MLTC

Syntax : $\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid \Diamond^*\varphi$;
where $p \in P$, a countable set of propositional variables. The Boolean connectives $\vee, \rightarrow, \leftrightarrow$ are defined as usual, and the duals $\Box\varphi$ and $\Box^*\varphi$ are defined as follows:

$\Box\varphi := \neg\Diamond\neg\varphi$; $\Box^*\varphi := \neg\Diamond^*\neg\varphi$.

$\Diamond\varphi$ is read as: there is a successor state where φ holds.

$\Diamond^*\varphi$ is read as: there is a reachable state where φ holds.

Semantics : A model is given by our usual Kripke model $M : (W, R, V)$. We now define 'φ holds in the pointed model (M, w)', denoted by : $M, w \models \varphi$.

* $\mathcal{M}, w \models p$ iff $w \in V(p)$
* $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
* $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$.
* $\mathcal{M}, w \models \Diamond\varphi$ iff there is some v in $\mathcal{M}$ with wRv and $\mathcal{M}, v \models \varphi$.
* $\mathcal{M}, w \models \Diamond^*\varphi$ iff there is some v in $\mathcal{M}$ with wR^*v and $\mathcal{M}, v \models \varphi$,

  R^*: reflexive, transitive closure of R.

Satisfiability and validity.

- A formula φ is satisfiable if there is a model $\mathcal{M} = (W, R, V)$ and a world w in $\mathcal{M}$, such that $\mathcal{M}, w \models \varphi$
- A formula φ is valid iff its negation is not satisfiable.

Some validities.

$$\left.\begin{array}{l}
- \ \Box^*\varphi \to (\varphi \wedge \Box\varphi) \\
- \ \Box^*\varphi \to \Box\Box^*\varphi \\
- \ (\varphi \wedge \Box\Box^*\varphi) \to \Box^*\varphi
\end{array}\right\} \quad \text{(H.W.)}$$

Thus, the following holds:

$$\Box^* \varphi \leftrightarrow (\varphi \wedge \Box\Box^* \varphi)$$

Informally, one can say that $\Box^* \varphi$ is a fixed point of the formula $\varphi \wedge \Box x$, or, a solution of the equation: $x \equiv \varphi \wedge \Box x$.

We now study various properties of MLTC, namely compactness, completeness, decidability.

Is MLTC compact?

$\{\Diamond^* p, \neg p, \neg \Diamond p, \neg \Diamond\Diamond p, \ldots\}$: An example of a fin-sat set of formulas, which is not satisfiable. Thus MLTC is a non-compact logic. Thus MLTC does not have generalised completeness ($\Gamma \vdash \varphi$ iff $\Gamma \models \varphi$).

Is MLTC weakly-complete ($\vdash \varphi$ iff $\models \varphi$)?

Yes. In contrast to the methodology followed earlier in this course, we will prove $\vdash \varphi$ iff $\models \varphi$ by giving the axiom system first.

Axiom system for MLTC :

Axioms : 1. Substitutional instances of propositional tautologies

2. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

3. $\Box^*(\varphi \to \psi) \to (\Box^*\varphi \to \Box^*\psi)$

4. $\Box^*\varphi \to (\varphi \wedge \Box\Box^*\varphi)$

Rules : 1. $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$ (M.P.) 2. $\dfrac{\vdash \varphi}{\vdash \Box\varphi}$ (Gen)

3. $\dfrac{\vdash \varphi}{\vdash \Box^*\varphi}$ ($\Box^*$-Gen) 4. $\dfrac{\vdash \varphi \to \Box\varphi}{\vdash \varphi \to \Box^*\varphi}$ (Ind)

H.W. Prove that the axiom system is sound.

Claim : The axiom system is complete.

Let us now prove this claim. As earlier we need to show that every consistent formula is satisfiable. (notice the difference in statement). Why ?

What we will show below is that every consistent formula is satisfiable in a model of some bounded size. So, in addition to getting completeness for MLTC we will also get strong finite model property, which will in turn give us decidability.

Claim: Every consistent formula is satisfiable

Proof: Let φ_0 be a consistent formula in MLTC. Let $Cl(\varphi)$ be the least set of formulas containing φ and closed under:

* $\neg\psi \in Cl(\varphi)$ iff $\psi \in Cl(\varphi)$
* if $\psi \wedge \chi \in Cl(\varphi)$, then $\{\psi, \chi\} \subseteq Cl(\varphi)$
* if $\Box\psi \in Cl(\varphi)$, then $\psi \in Cl(\varphi)$
* if $\Box^*\psi \in Cl(\varphi)$, then $\{\psi, \Box\Box^*\psi\} \subseteq Cl(\varphi)$

Note: By identifying $\neg\neg\psi$ with ψ, we restrict ourselves to a finite set $Cl(\varphi)$, where,

$$|Cl(\varphi)| = O(|\varphi|)$$

Down-closed sets

A set A of formulas is said to be down-closed if:

- if $\neg\varphi \in A$, then $\varphi \notin A$
- if $\varphi \wedge \chi \in A$, then $\{\varphi, \chi\} \subseteq A$
- if $\varphi \vee \chi \in A$, then $\varphi \in A$ or, $\chi \in A$
- if $\Box^*\varphi \in A$, then $\{\varphi, \Box\Box^*\varphi\} \subseteq A$.
- if $\Diamond^*\varphi \in A$, then $\varphi \in A$ or, $\Diamond\Diamond^*\varphi \in A$.

Let D_φ denote the set of all down-closed subsets of $Cl(\varphi)$.

What would be the Kripke model?

Consider any formula φ. We define the following

Tableau graph

The tableau graph of φ, denoted by $G_\varphi =$ (D_φ, R_φ), where: $(A, B) \in R_\varphi$ iff

$\{\psi \mid \Box\psi \in A\} \subseteq B$.

Note: If $\Box^*\psi \in A$, then both $\psi, \Box^*\psi \in B$ (follows from Axiom 4)

What properties should this graph satisfy?

We attempt to get a 'model graph' that satisfies the different conditions that we need corresponding to the modal formulas

Model graph

Take $\mathcal{U} \subseteq D_\varphi$. The subgraph induced by $\mathcal{U}$ is said to be a model graph if:

#1. there is $A \in \mathcal{U}$, such that $\varphi \in A$

#2. for all $A \in \mathcal{U}$, if $\Diamond\psi \in A$, then there exists $B \in \mathcal{U}$, such that $(A, B) \in R_\varphi$ and $\psi \in B$.

#3. for all $A \in \mathcal{U}$, if $\Diamond^*\psi \in A$, there exists a path in $\mathcal{U}$ from A to B such that $\psi \in B$.

Let us now consider the following:

What is exactly the Kripke model at hand?

Define a valuation function $V_\varphi : \wp \rightarrow 2^{D_\varphi}$ as follows: $V_\varphi(p) = \{A : p \in A\}$.

Thus we have a model $M_\varphi : (D_\varphi, R_\varphi, V_\varphi)$