# Lecture 28

Given the model, we have to find a model graph $\mathcal{U}$ and prove the following:

Truth lemma: For all $\psi \in Cl(\varphi)$, $A \in \mathcal{U}$, $M_\varphi, A \models \psi$ iff $\psi \in A$.

If we assume the lemma, given a formula $\varphi_0$ we can construct the model $M_{\varphi_0}: (D_{\varphi_0}, R_{\varphi_0}, V_{\varphi_0})$, and consider an $A_0 \in \mathcal{U}$, such that $\varphi_0 \in A_0$. Then we have: $M_{\varphi_0}, A_0 \models \varphi_0$.

So, $\varphi_0$ is satisfiable. The size of the model $M_{\varphi_0}$ depends on the size of the formula $\varphi_0$.

So, all we need to do is to get to a suitable model graph, a subgraph of $G_{\varphi_0}$.

That will take care of the modal formulas in $Cl(\varphi_0)$. For the boolean formulas, the proof of truth lemma is as usual. We will only concentrate on the modal cases.

Proof of the lemma:

We start with considering maximal consistent subsets (MCS) of $Cl(\varphi_0)$ which we name atoms. Let $AT_{\varphi_0}$ denote the set of all atoms of $Cl(\varphi_0)$. Evidently, every atom is a down-closed subset of $Cl(\varphi_0)$.

Then it suffices to show that the subgraph induced by $AT_{\varphi_0}$ is a model graph. Now, since $\varphi_0$ is a consistent formula in $Cl(\varphi_0)$, it can be extended to an MCS $A_0$ in $AT_{\varphi_0}$ with $\varphi_0 \in A_0$.

Thus the condition ($\sharp 1$) of a model graph is satisfied. Let us now consider condition ($\sharp 2$). Let $A$ be an atom and $\Diamond\psi \in A$. Then, by using the axioms and rules of basic modal logic

PML (Axioms (1), (2), Rules (1), (2)), we can show that there exists an atom $B$ such that $(A, B) \in R_{\varphi_0}$ and $\psi \in B$. So, we are done (Check!)

Finally, we try to check condition (#3).

Consider an atom $A$ such that $\Diamond^* \psi \in A$. We need to find an atom $B$ reachable from $A$ such that $\psi \in B$. Since, $\Diamond^* \psi \in A$, either $\psi \in A$, or, $\Diamond \Diamond^* \psi \in A$ (as, $A$ is down-closed). In the former case, we are done. In the latter case, we have an atom $C$ accessible to $A$ with $\Diamond^* \psi \in C$. We can continue this process and get a path starting from $A$.

What is the guarantee that this process will end finitely and an atom $B$ will be reached with $\psi \in B$?

We show this by using the Ind-rule.

Let us first introduce some notations noting that we are in the realm of finite sets.

* Let $A \in AT_{\varphi_0}$. $\hat{A}$ denotes the conjunction of formulas in $A$.
* Let $X \subseteq AT_{\varphi_0}$. $\overline{X}$ denotes $\bigvee_{A \in X} \hat{A}$.

Then, we have the following:

- if $\varphi \in A$, $\vdash \hat{A} \to \varphi$
- if $X \subseteq AT_{\varphi_0}$ and $A \in X$, $\vdash \hat{A} \to \overline{X}$
- $\vdash \overline{AT_{\varphi_0}}$ (Why?)
- if $X$ and $Y$ partitions $AT_{\varphi_0}$, $\vdash \overline{X} \leftrightarrow \neg \overline{Y}$ (Why?)

With these notations out of the way, let us come back to the main argument. Let $\Diamond^* \psi \in A$. Let $\mathcal{H}$ denote the set of all atoms reachable from $A$ in the induced subgraph of $AT_{\varphi_0}$. If there exists $B$ in $\mathcal{H}$ with $\psi \in B$, we are done. Suppose not. Then, for all $B \in \mathcal{H}$, $\neg\psi \in B$. So, $\vdash \overline{\mathcal{H}} \to \neg\psi$

(Check!) Then, $\vdash \Box^* \overline{\mathcal{H}} \to \Box^* \neg\psi$ ($\Box^*$-Gen rule and Axiom 3). We now claim the following.

Claim: $\vdash \overline{\mathcal{H}} \to \Box \overline{\mathcal{H}}$

Suppose we have the claim. Then by Ind-rule, we have $\vdash \overline{\mathcal{H}} \to \Box^* \overline{\mathcal{H}}$. So, we have: $\vdash \overline{\mathcal{H}} \to \Box^* \neg\psi$. Now, as $A \in \mathcal{H}$, $\vdash \hat{A} \to \overline{\mathcal{H}}$.

So, $\vdash \hat{A} \to \Box^* \neg\psi$. But $\vdash \hat{A} \to \Diamond^* \psi$, as, $\Diamond^* \psi \in A$. Thus we arrive at a contradiction. Hence the result, that is, whenever $\Diamond^* \psi \in A$, there is an atom $B \ni \psi$, and a path from $A$ to $B$.

Proof of the claim:

To show that $\vdash \overline{\mathcal{H}} \to \Box \overline{\mathcal{H}}$. Suppose not. Then, $\overline{\mathcal{H}} \wedge \Diamond \neg \overline{\mathcal{H}}$ is consistent. (Why?)

Let $\mathcal{K} = AT_{\varphi_0} \setminus \mathcal{H}$. We have that $\overline{\mathcal{H}} \wedge \Diamond \overline{\mathcal{K}}$ is consistent (Why?). So, for some $B \in \mathcal{H}$ and $C \in \mathcal{K}$, we have: $\hat{B} \wedge \Diamond \hat{C}$ is consistent (Why?). So, we have that $(B, C) \in R_{\varphi_0}$.

H.W. $(B, C) \in R_{\varphi_0}$ iff $\hat{B} \wedge \Diamond \hat{C}$ is consistent.

So, $C$ is reachable from $B$, and hence $C$ is reachable from $A$. Thus we have that $C \in \mathcal{H}$, a contradiction, as $C \in \mathcal{K}$. So, we have our claim. This completes the proof.

So, we have that every consistent formula is satisfiable, which in turn gives our weak completeness theorem for MLTC. The model we just constructed is finite and bounded by $2^{O(|\varphi|)}$. Thus we have strong finite model property for MLTC, which shows that satisfiability problem for MLTC is decidable. Hence, MLTC is weakly complete and decidable.

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