Lecture 3

Bound variable and free variables

Bound variables : Variables which occur under the scope of a quantifier.

Free variables : Variables that are not bound.

$\forall x \exists y \, P_1^2 x y$: Both x and y are bound.

$\forall x \, P_1^2 x y$: x is bound, y is free.

Set of free variables occuring in terms :

$FV(x) = \{x\}$.

$FV(c) = \Phi$

$FV(f_i^n t_1 t_2 \dots t_n) = \bigcup_{k=1}^{n} FV(t_k)$

Set of free variables occuring in formulas

$$FV(t_1 = t_2) = FV(t_1) \cup FV(t_2)$$

$$FV(p_i^n\, t_1 t_2 \ldots t_n) = \bigcup_{k=1}^{n} FV(t_k)$$

$$FV(\neg\varphi) = FV(\varphi)$$

$$FV(\varphi \wedge \psi) = FV(\varphi) \cup FV(\psi)$$

$$FV(\varphi \vee \psi) = FV(\varphi) \cup FV(\psi)$$

$$FV(\varphi \rightarrow \psi) = FV(\varphi) \cup FV(\psi)$$

$$FV(\varphi \leftrightarrow \psi) = FV(\varphi) \cup FV(\psi)$$

$$FV(\forall x\, \varphi) = FV(\varphi) \setminus \{x\}$$

$$FV(\exists x\, \varphi) = FV(\varphi) \setminus \{x\}.$$

Proposition: Let φ be a formula and (D, I) be a structure. For any two assignment functions g_1, g_2, if g_1 and g_2 agree on $FV(\varphi)$, we have: $(D, I, g_1) \models \varphi$ iff $(D, I, g_2) \models \varphi$.

Size of a formula: The number of connectives and quantifiers present in the formula.

Proof of the proposition:

We prove this result by applying induction on the size of the formula φ. We have a domain D and an interpretation I. We also have two variable assignment functions g_1 and g_2 which agree on the free variables of φ. To prove:

$M_1 = (D, I, g_1) \models \varphi$ iff

Base Case: (i) $\varphi : t_1 \equiv t_2$: Then, $M_1 \models \varphi$ iff $M_1 \models t_1 \equiv t_2$ iff $g_1(t_1) = g_1(t_2)$ iff $g_2(t_1) = g_2(t_2)$ iff $M_2 \models t_1 \equiv t_2$ iff $M_2 \models \varphi$.

(ii) $\varphi : p_i^n t_1 t_2 \cdots t_n$: Then, $M_1 \models \varphi$ iff $M_1 \models p_i^n t_1 t_2 \cdots t_n$ iff $(g_1(t_1), \ldots, g_1(t_n)) \in I(p_i^n)$

iff $(g_2(t_1), \ldots, g_2(t_n)) \in I(p^n_i)$ iff

$\mathcal{M}_2 \models p^n_i t_1 t_2 \cdots t_n$ iff $\mathcal{M}_2 \models \varphi$.

Induction Hypothesis: Suppose the result holds for all formulas with size $\leq m$.

Induction Step: Suppose the size of φ is $m+1$.

Case I: $\varphi : \neg\psi$: $\mathcal{M}_1 \models \varphi$ iff $\mathcal{M}_1 \models \neg\psi$ iff $\mathcal{M}_1 \not\models \psi$ iff $\mathcal{M}_2 \not\models \psi$ (by I.H.) iff $\mathcal{M}_2 \models \neg\psi$ iff $\mathcal{M}_2 \models \varphi$.

Case II: $\varphi : \psi \wedge \chi$: $\mathcal{M}_1 \models \varphi$ iff $\mathcal{M}_1 \models \psi \wedge \chi$ iff $\mathcal{M}_1 \models \psi$ and $\mathcal{M}_1 \models \chi$ iff $\mathcal{M}_2 \models \psi$ and $\mathcal{M}_2 \models \chi$ (by I.H.) iff $\mathcal{M}_2 \models \psi \wedge \chi$ iff $\mathcal{M}_2 \models \varphi$.

Case III: $\varphi : \forall x \psi$. $\mathcal{M}_1 \models \varphi$ iff $\mathcal{M}_1 \models \forall x \psi$

iff for all $d \in D$, $\mathcal{M}_{1[x \to d]} \models \psi$ iff

$\mathcal{M}_{2[x \to d]} \models \psi$ for all $d \in D$ iff $\mathcal{M}_2 \models \forall x \psi$

iff $\mathcal{M}_2 \models \varphi$.

To prove iff: Take any $d \in D$. Then

we have: $$\ell_{k[x \to d]}(y) = \begin{cases} \ell_k(y), & y \neq x \\ d, & y = x \end{cases}$$

$k = 1, 2$.

Also, $FV(\varphi) = FV(\psi) \setminus \{x\}$.

We will be able to prove iff if

we can show that $\ell_{1[x \to d]}$ and $\ell_{2[x \to d]}$

agree on $FV(\psi)$. Because then, by I.H.

we will have: $\mathcal{M}_{1[x \to d]} \models \psi$ iff $\mathcal{M}_{2[x \to d]} \models \psi$.

Now, $g_{1[x \to d]}$ and $g_{2[x \to d]}$ agree on $FV(\psi)$ as (i) they agree on $FV(\varphi)$, and (ii) they agree on x. We note that $FV(\varphi) = FV(\psi) \setminus \{x\}$. Thus we have:

$\mathcal{M}_{1[x \to d]} \models \psi$ iff $\mathcal{M}_{2[x \to d]} \models \psi$.

Since we have taken an arbitrary $d \in D$, we have that for all $d \in D$, $\mathcal{M}_{1[x \to d]} \models \psi$ iff $\mathcal{M}_{2[x \to d]} \models \psi$. Thus, $\mathcal{M}_1 \models \forall x \psi$ iff $\mathcal{M}_2 \models \forall x \psi$, i.e., $\mathcal{M}_1 \models \varphi$ iff $\mathcal{M}_2 \models \varphi$.

This completes the proof. $\blacksquare$

Why did not we do the case for $\varphi \vee \psi$, $\varphi \to \psi$, $\varphi \leftrightarrow \psi$, $\exists x \varphi$?

Sentences:

A formula which does not have a free variable.

Example: $\forall x\,(P^1_1 x)$, $\forall x\,\exists y\,(x = y)$.

Corollary: Let φ be a sentence and (D, I) be a structure. Then, either $(D, I, g) \models \varphi$ for all assignments g or, $(D, I, g) \not\models \varphi$ for all assignments g.

H.W. Prove this corollary.

When φ is a sentence, we have:

φ is true in a structure (D, I)

$$\big((D, I) \models \varphi \big)$$

OR

φ is false in the structure (D, I)

$$\big((D, I) \models \varphi \big)$$

Expressivity of first-order language.

Consider a first-order language $\mathcal{L}$, s.t. $\mathcal{C}_{\mathcal{L}} = \mathcal{F}_{\mathcal{L}} = \mathcal{P}_{\mathcal{L}} = \emptyset$. For this discussion, let us assume D can be empty as well.

With such a language we can only talk about sets (as we cannot put structure on these sets).

- $\forall x \neg (x \equiv x)$: empty set
- $\forall x \forall y (x \equiv y)$: all sets with ≤ 1 element
- $\exists x \forall y (x \equiv y)$: singleton sets
- $\exists x (x = x) \wedge \forall x \forall y (x = y)$: singleton sets.
- $\exists x \exists y (\neg (x = y) \wedge \forall z (z = x \vee z = y))$: sets containing exactly two elements.

- $\exists x \exists y \exists z (\neg(x=y) \wedge \neg(y=z) \wedge \neg(x=z)$
$\wedge \forall w (w=x \vee w=y \vee w=z))$.

sets containing exactly three elements.

Similarly, we can express sets having exactly k elements, for any finite number k.

Can we express sets containing infinitely many elements? (Think about it!)

Notes:

- Here, we were only considering sentences.
- They are either true or false in a structure
- We generally say: a formula φ is expressing a class of structures $\mathcal{C}$, say, if the following holds: $M \models \varphi$ iff $M \in \mathcal{C}$.
- We will deal with these concepts in more detailed manner in the next class.