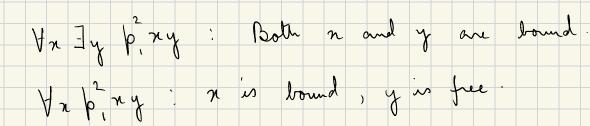
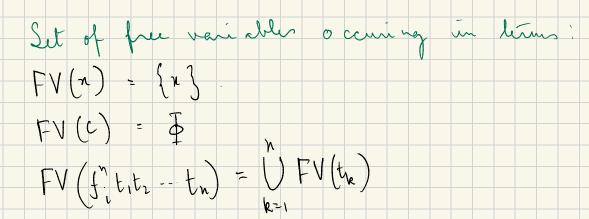


Bound vari able and free variables Bound variables : Variables which occur under the scope of a quantifier. Free variables : Variables that are not

bound .





Set of free variables occuring in formulas  $FV(t_1 = t_2) = FV(t_1) \cup FV(t_2)$  $FV\left(p_{i}^{n}t_{1}t_{2}-t_{n}\right)=\bigcup_{k=1}^{n}FV\left(t_{k}\right)$  $FV(\tau q) = FV(q)$ FV (q N y) - FV (q) U FV (y) FV(qVy) = FV(q)VFV(y) $FV(q \rightarrow \psi) = FV(q) \cup FV(\psi)$  $FV(q \leftrightarrow \psi) = FV(q) U FV(\psi)$  $FV(\forall x \varphi) = FV(\varphi) \setminus \{x\}$  $FV(\exists nq) = FV(q) \setminus \{n\}$ . Proposition: Let q be a formula and (D, I) be a structure For any two assignment functions y, y, y g, and y ague on FV(q), we have  $(D, \overline{J}, \overline{Y}, \overline{Y}) \neq \varphi$  iff  $(D, \overline{J}, \overline{Y}, \overline{Y}) \neq \varphi$ 

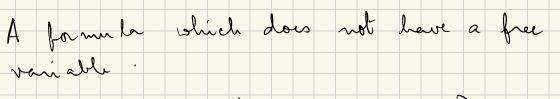
Size of a formula: The number of connectives and quantifiers present in the formula. Proof of the proposition : We prove this result by applying induction on the size of the formula q ble have a domain D and an interpretation I ble also have two variable assignment functions by, and by which agree on the prec variables of Q. To prove!  $M_1 = (D, L, M_1) F q M_1$ Base Case: (i) q: t, = tz: Thun, M, F q ff  $M_1 \neq t_1 \equiv t_2$  iff  $y_1(t_1) = y_1(t_2)$  iff  $y_2(t_1) = y_2(t_2)$  if  $M_2 \neq t_1 \equiv t_2$  if  $M_2 \neq \varphi$ . (ii) q:  $p_i$ t,  $t_2$ -...t\_n: Jhen,  $M_i \neq q$  iff  $\mathcal{M}_{i} \models p^{n}_{i} t_{i} t_{2} \dots t_{n}$  if  $(y_{i}(t_{i}), \dots, y_{i}(t_{m})) \in I(p^{n}_{i})$ 

 $\mathcal{H}\left(\mathcal{Y}_{2}\left(t_{1}\right),\ldots,\mathcal{Y}_{2}\left(t_{n}\right)\right)\in\mathbb{E}\left(p_{1}^{n}\right)\quad \mathcal{H}$  $\mathcal{M}_2 \models p_i^n t_i t_2 \cdots t_n \quad \mathcal{M}_k \quad \mathcal{M}_2 \models q$ Induction Hypothesis : Suppose the result holds for all formulas with size & m. Induction Step: Suppose the size of q in mtl. Case I: Q: 7Y: M, FQ iff M, F7Y M M, H Y IH M2 H Y (by [.H.) if M2 F7Y M M2 FQ. Care II Q: YAX! M, FQ M M, FYAX M M, FY and M, FX M M2FY and M2FX (by I.H.) if MZ FYAX if MZ FQ Core III Q: Vay Mitq iff Mittry

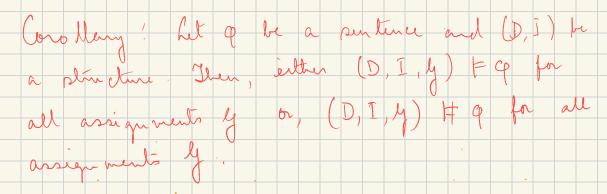
if for all dED, MIGAN FY M2[x)d] FY for all dED iff MEFtry  $M_2 \models \varphi$ To prove iff . Take any dED. Then we have :  $y_k[x \rightarrow d](y) = (y_k(y), y \neq x)$  $(\lambda, y=n)$ k = 1,2. Also, FV(q) = FV(y) \ {z}. be will be able to prove iff if we can show that Gigard and Gigard aque on FV(V). Because Then, by I.H. we will have : M, EY iff M2 (2+3) FV.

Now, Giand Jacard Jagee on FV(4) as (i) they agree on FV(q), and (ii) they agree on n ble note that FV(q) = FV(y) \ {x]. Ilms we have: MIENTAJ FY MA MIENTAJ FY. Since we have taken an arbitrary  $d \in D$ . we have that for all  $d \in D$ ,  $M_1 \xrightarrow{FP}$  $iH = M_2 \xrightarrow{[n-d]} F = Y$ . Jhuns,  $M_1 \xrightarrow{F} = V \xrightarrow{P} = V$ .  $M_1 = M_2 \neq \forall x \psi$ , i.e.,  $M_1 \neq \varphi$   $M_1 \neq \varphi$ . This completes the proof. Ø Why did not we do the case for  $qV\Psi$ ,  $q \rightarrow \Psi$ ,  $q \leftarrow \Rightarrow \Psi$ ,  $\exists x \varphi$ ?

Sentences:



Example:  $\forall n(P', n), \forall n \exists y(n = y)$ 



H.W. Prove this cono lary -

When q is a sentence, we have q'i true in a structure (D, I)  $((D,L) \models q)$ q in false in the structure (D,1) $((D,1) \neq q)$ 

Expressivity of first-order language Consider a first-order longuage h, s.t. ly = J\_L = P\_L = & For this discussion, let us assume D can be empty as will. With such a language we can only talk about sets (as we cannot fent structure on these sets). - Vn 7 (n = n) : empty set - Yz Yy (z = y) : all sets with < L element Zr Vy (n=y): singleton sets - Zn (n=n) A Hn Hy (n=y) : singleton sets. - JrJy (7(n=y) N U2(2=2V2=y)): sets containing exactly two elements

 $\exists x \exists y \exists z (\tau(x=y) \land \tau(y=z) \land \tau(x=z))$  $\Lambda \forall \omega (\omega = x \vee \psi = y \vee \omega = z)$ rets containing exactly three elements Similarly, we can express sets having enaethy k elements, for any firste number k Com we enpress sets containing infinitely many elements ? (Think about it!) Notes ! - Here, we were only considering sentences. - They are either true or false in a structure - We generally say: a formula q is expressing a class of structures &, say, if the follo ving holds: MEQ iff MER. De toll deal joth these concepts in more detailed manner in the next class.