Lecture 4

What FOL formulas can express and what FOL formulas cannot enpress? Definable relations in structures Let h be a first-order danguage and let A be an h-structure (A: (D, I)). An n-any relation R on D is said to be definable in A if there is an h-formula q, say, whose free variables are $\kappa_1, \kappa_2, \cdots, \kappa_n$, ranch that : $(a_1, a_2, \dots, a_n) \in \mathbb{R}$ if $A_{[x_i \rightarrow a_1, x_2 \rightarrow a_2, \dots, x_n \rightarrow a_n]} \neq \varphi(x_i, \dots, x_n \rightarrow a_n]$ for all a, az, ..., an ED. Examples :

, + , •) $1 \quad \& \quad (IR, 0, 1)$ $\varphi(\mathbf{x}) := \exists y(\mathbf{x} = y \cdot y)$ (a) P = {a: a>0} (b) $\leq = \{(a,b): a \leq b\} \quad q(n,\gamma):= \exists z(\gamma = n + z \cdot z)$ $\{(a,b), (a,c)\}$ 2. $G = (\{a, b, c\},$ (edge relation : E) (a) {b,c} $\phi(r) = 32Ezr$ (b) {b} Not definable (no way to distinguish {b} and {c} Definatility of a class of structures het K be a class of structures. K is said to be definable in an FOL language h if there is a sentence, o, say, such that i







if for all f.o - formulas \$\$, M, EQ $M_2 \neq Q -$ Lu fact, we would be more interested in the notion of : Elementary equivalence Two structures A = (DA, FA) and B = (DB, IB) are said to be elemen tanky equivalent if for all fosentences T, AFT iff BFT We say A = 83. Proposition i If two structures are is on on place, then they are elementaily equivalent Before proving this proposition, let

us fin at de fine homomon plus m bet ween two structures. L (L, F, F) be gwen Let A and B be two L-structures A homomorphism h of A into B is a function h: DA DB n.t. (i) $h(I_{A}(c)) = I_{b}(c)$ for all $c \in \mathcal{L}$ $(ii) M(f(a_1, \dots, a_n)) = f(h(a_1), \dots, h(a_n))$ for all n-any function symbols, (111) $(a_1, \dots, a_n) \in p$ iff $(h(a_1), \dots, h(a_n)) \in p_B$ $\mu all a_1, \dots, a_n \in D_{\mathbf{A}}$ In addition, if h is injustive, h is said to be an embedding of A

into B. Moreover, if h is surjective, then A is said to be isomorphic to B. het if and to be two h-structures and he be a homomorphism. How would the terms behave under In ? Let y be an assignment of variables in A. Then for any term t , $h(y_{\mathcal{F}}(t)) = y_{\mathcal{B}}(t),$ where $y_{\mathcal{B}} = h \circ y_{\mathcal{A}}$ H.W. Prove the statement above. Theorem: Let A, B, Y, YB are given as above. (i) For quantifier pre for mulas q, without equality, we have i $(A, Y) \neq \varphi \neq (B, Y) \neq \varphi$

Proof idea: We prove this by induction on the singe of q Derse carse i Q - pt, tz - . tr $(A, Y_A) \models \varphi$ $iii (A, y_A) \models p^n t_1 - ... t_n$ $\mathcal{H}\left(\mathcal{Y}_{A}(t_{i}),\ldots,\mathcal{Y}_{A}(t_{n})\right)\in\mathcal{F}_{A}$ $\mathcal{M}\left(h\left(y_{A}(t_{i})\right), \dots, h\left(y_{A}(t_{n})\right)\right) \in \beta_{\mathcal{B}}^{n}$ $\mathcal{M}\left(\mathcal{Y}_{\mathcal{B}}(t_{1}), -\cdots, \mathcal{Y}_{\mathcal{B}}(t_{n})\right) \in \mathfrak{p}_{\mathcal{B}}^{n}$ \mathcal{M} ($\mathcal{B}, \mathcal{Y}_{\mathcal{B}}$) $\models p^{n} t_{1} \cdots t_{n}$ $\mathcal{H}(\mathcal{B},\mathcal{Y}_{\mathcal{B}}) \models \varphi$ Induction Hypothesis Suppose the result hold for all formular of size < n.

Induction Step $q := \gamma \gamma ; q := \gamma \Lambda \chi$ (Complete the proof !) This completes the proof of (i). (ii) Consider quantifier per formulas with equality [Assume h is injective] Proof idea: (A, MA) = t1 = t2 $\mathcal{A} = \mathcal{A} =$ (: h is injustive) $\mathcal{H}_{h} \wedge (\mathcal{Y}_{A}(t_{1})) = h (\mathcal{Y}_{A}(t_{2}))$ $\mathcal{M} \quad \mathcal{Y}_{\mathcal{B}}(t_1) = \mathcal{Y}_{\mathcal{B}}(t_2)$ $\mathcal{H}(\mathcal{B},\mathcal{H}_{\mathcal{B}}) \models t_1 \equiv t_2$ (iii) Consider quantified formular (q=] x y) [Assume the is surjective, in addition]

 $(A, YA) \neq \exists a \forall$ Proof : some dED = $(A, Y_{A}(n) \neq Y f n)$ ([.H.) $\Rightarrow (B, H_{B(a)}) \neq Y$ \Rightarrow (β , ψ_0) $\neq \exists n \psi$. Conversely, (B, 40) F Iny => (8, 4 b [2+) d']) FY for some d'EDs for some dEDA => $(8, 40 [n \rightarrow h(d)]) \neq \psi$ =) $(A, Y_A(n \rightarrow d)) \models \gamma$ (Γ, H) => (A, YA) F Jar This completes the proof for all formulas What enactly did we prove here ?