LECTURE 4

17.01.2024

Corollary: Let φ be a sentence and (D, I) be a structure. Then, either for all assignment functions g, $(D, I, g) \models \varphi$ or for all g, $(D, I, g) \not\models \varphi$.

H.W. Prove it.

— When φ is a sentence, we say:

φ is true in a structure (D, I),

$$\left[(D, I) \models \varphi \right]$$

OR

φ is false in a structure (D, I),

$$\left[(D, I) \not\models \varphi \right]$$

On expressivity

Consider a first order language $\mathcal{L}$, s.t. $\mathcal{C}_{\mathcal{L}} = \mathcal{F}_{\mathcal{L}} = \mathcal{P}_{\mathcal{L}} = \Phi$. For this discussion, let us assume that D can be empty as well. With such a language we can only talk about sets (as we cannot put any structure on these sets).

- $\forall x \neg (x = x)$: empty set.

- $\forall x \forall y (x = y)$: all sets whose cardinality is ≤ 1.

- $\exists x \forall y (x = y)$
- $\exists x (x = x) \wedge \forall x \forall y (x = y)$

$\Big\}$: singleton sets

- $\exists x \exists y \left(\neg (x = y) \wedge \forall z (z = x \vee z = y) \right)$:

sets containing exactly 2 elements

- $\exists x \exists y \exists z \left(\neg(x = y) \wedge \neg (y = z) \wedge \neg (z = x) \wedge \forall w (w = x \vee w = y \vee w = z) \right)$

sets containing exactly 3 elements

Similarly, we can express sets having exactly k elements, for any finite k.

Can we express sets containing infinitely many elements?

Please think about it !!

(Semantic) Consequence relation :

Γ : a set of formulas

φ : a formula

$\boxed{\vDash \subseteq \wp(\mathcal{L}) \times \mathcal{L}}$

We say that φ is a semantic consequence of Γ (denoted by $\Gamma \vDash \varphi$), if for all models $\mathcal{M}$, $\underbrace{\mathcal{M} \vDash \gamma \text{ for all } \gamma \in \Gamma}_{\mathcal{M} \vDash \Gamma}$ imply $\mathcal{M} \vDash \varphi$.

Example : $\Gamma = \{P^1 x \to Q^1 x,\ Q^1 x \to R^1 x\}$

$$\varphi = P^1 x \to R^1 x$$

To show : $\Gamma \vDash \varphi$

We need to show : for all models $\mathcal{M}$, if $\mathcal{M} \vDash \Gamma$ then $\mathcal{M} \vDash \varphi$.

Take any model $\mathcal{M} : (D, I, g)$

Suppose $\mathcal{M} \models \Gamma'$, that is, suppose $\mathcal{M} \models P'x \to Q'x$ and $\mathcal{M} \models Q'x \to R'x$

To show : $\mathcal{M} \models P'x \to R'x$.

Suppose $\mathcal{M} \models P'x$. To show : $\mathcal{M} \models R'x$.

Since, $\mathcal{M} \models P'x \to Q'x$, we have $\mathcal{M} \models Q'x$. Now, since, $\mathcal{M} \models Q'x \to R'x$, $\mathcal{M} \models R'x$. This completes the proof. $\square$

Now, suppose that $\Gamma = \Phi$. Then, when we have $\Gamma \models \varphi$, we basically have $\Phi \models \varphi$ (denoted by $\models \varphi$), which says 'φ is satisfied by all models'.

- A formula φ is said to be <u>valid</u> if for every model $\mathcal{M}$, $\mathcal{M} \models \varphi$
- A formula φ is said to be <u>satisfiable</u> if there is a model $\mathcal{M}$, s.t. $\mathcal{M} \models \varphi$

Examples :

Consider a f.o.l. $\mathcal{L}$ with $\mathcal{C} = \phi$, $\mathcal{F} = \phi$ and $\mathcal{P} = \{P^2\}$. Consider a structure (D, I), D being a non-empty

$I_1(P^2) = D \times D \longleftrightarrow \varphi_1 : \forall x \forall y\, P^2 xy$

$I_2(P^2) = \phi \longleftrightarrow \varphi_2 : \forall x \forall y\, \neg P^2 xy$

$I_3(P^2) =$ a serial relation $\longleftrightarrow \varphi_3 : \forall x \exists y\, P^2 xy$

$\Downarrow$

(for each $d \in D$, there is $d' \in D$ s.t. $(d, d') \in I_3(P^2) = R$, say)

Examples of valid and satisfiable formulas

- $\forall x \forall y\, (P^2 xy \vee \neg P^2 xy)$: valid
- $\forall x \forall y\, (P^2 xy \wedge \neg P^2 xy)$: true in empty models only
- $\exists x \exists y\, (P^2 xy \wedge \neg P^2 xy)$: unsatisfiable

H. W.

1. Let $\Gamma = \{\varphi, \neg \varphi\}$. Then show that $\Gamma \models \psi$ for all formulas ψ.

2. Show that if $\varphi \in \Gamma$, then $\Gamma \models \varphi$.

3. Let φ be a formula. Then, φ is valid iff $\neg\varphi$ is not satisfiable: Prove or disprove

4. If $\Gamma_1 \subseteq \Gamma_2$ and $\Gamma_1 \models \varphi$, then show that $\Gamma_2 \models \varphi$.

Proof ideas are discussed in class.