LECTURE 5

24.01.2024

The use/abuse of the symbol $\models$.

$\mathcal{M} \models \varphi$ (satisfies)

$\Gamma \models \varphi$ (entails)

Models and Theories

Given any formula φ, define $\mathrm{Mod}(\varphi) = \{\mathcal{M} \mid \mathcal{M} \models \varphi\}$. Given a set of formulas Γ, define $\mathrm{Mod}(\Gamma) = \{\mathcal{M} \mid \mathcal{M} \models \Gamma\}$.

Let $\mathcal{K}$ be a class of models. The theory of $\mathcal{K}$, denoted by $\mathrm{Th}(\mathcal{K})$, is defined by $\mathrm{Th}(\mathcal{K}) = \{\varphi \mid \mathcal{M} \models \varphi$ for all models $\mathcal{M}$ in $\mathcal{K}\}$.

Let Γ_1, Γ_2 be two sets of formulas s.t. $\Gamma_1 \subseteq \Gamma_2$. Then,

$$\mathrm{Mod}(\Gamma_1) \supseteq \mathrm{Mod}(\Gamma_2)$$

Let $\mathcal{K}_1, \mathcal{K}_2$ be two classes of models s.t. $\mathcal{K}_1 \subseteq \mathcal{K}_2$. Then,

$$\mathrm{Th}(\mathcal{K}_1) \supseteq \mathrm{Th}(\mathcal{K}_2)$$

Consequence of a set of formulas

Let Γ be a set of formulas. The consequence of Γ, denoted by $\mathrm{Con}(\Gamma)$ is defined as $\mathrm{Con}(\Gamma) = \mathrm{Th}(\mathrm{Mod}(\Gamma))$.

Proposition: Let Γ be a set of formulas and φ be a formulas. We have: $\varphi \in \mathrm{Con}(\Gamma)$ iff $\Gamma \models \varphi$.

Proof: Suppose $\Gamma \models \varphi$. To show: $\varphi \in \mathrm{Con}(\Gamma)$. Now, all models in $\mathrm{Mod}(\Gamma)$ satisfy φ. So, $\varphi \in \mathrm{Th}(\mathrm{Mod}(\Gamma))$, that is, $\varphi \in \mathrm{Con}(\Gamma)$.

Conversely, suppose that $\varphi \in \mathrm{Con}(\Gamma)$ To show: $\Gamma \models \varphi$. Now, $\varphi \in \mathrm{Th}(\mathrm{Mod}(\Gamma))$

implies that $\mathcal{M} \models \varphi$ for all $\mathcal{M} \in Mod(\Gamma)$.

Thus, $\Gamma \models \varphi$.

This completes the proof.

H.W: Properties of $Con(\Gamma)$.

1. $\Gamma \subseteq Con(\Gamma)$.
2. if $\Gamma_1 \subseteq \Gamma_2$, then $Con(\Gamma_1) \subseteq Con(\Gamma_2)$
3. $Con(Con(\Gamma)) = Con(\Gamma)$.

What does it mean to say that $\Gamma \not\models \varphi$?

$\Gamma \not\models \varphi$ iff there is a model $\mathcal{M}$, s.t. $\mathcal{M} \models \Gamma$ and $\mathcal{M} \not\models \varphi$

iff there is a model $\mathcal{M}$, s.t. $\mathcal{M} \models \Gamma$ and $\mathcal{M} \models \neg\varphi$

iff there is a model $\mathcal{M}$, s.t. $\mathcal{M} \models \Gamma \cup \{\neg\varphi\}$

iff $\Gamma \cup \{\neg\varphi\}$ is satisfiable.

More on satisfiability.

Let Γ be a set of formulas, φ be a formula.

- φ is satisfiable if there is a model of φ.
- Γ is satisfiable if there is a model of Γ.

Examples

- Consider an f.o.l $\mathcal{L}$, with parameters $P^1_1, P^1_2, P^1_3, \ldots$

- $\Gamma = \{P^1_1 x\}$. Is Γ satisfiable?

To answer in the affirmative, we have to find a model (D, I, g) s.t. $(D, I, g) \models P^1_1 x$.

① $D = \{a, b\}$. $I(P^1_1) = \{a\}$, $g : V \to D$: $g(y) = a$ for all y.

Then, $(D, I, g) \models P^1_1 x$, as $g(x) \in I(P^1_1)$.

② $D = \{a, b\}$. $I(P^1_1) = \{a\}$, $g : V \to D$: $g(y) = a$, $y = x$
b, $y \neq x$

Then, $(D, I, g) \models P^1_1 x$, as $g(x) \in I(P^1_1)$.

③ $D = \mathbb{N}$. $I(P^1_1) = \{n \in \mathbb{N} \mid n \text{ is even}\}$.

$g : V \to D$: $g(y) = 2$ if $y = x$
0 if $y \neq x$.

Then, $(D, I, g) \models P^1_1 x$, as $g(x) \in I(P^1_1)$.

Thus, Γ is indeed satisfiable.

- $\Gamma = \{P^1_1 x, \neg P^1_1 x\}$; unsatisfiable.

- $\Gamma = \{P^1_1 x, P^1_2 x, \neg(P^1_1 x \wedge P^1_2 x)\}$: unsatisfiable

- $\Gamma = \{P^1_1 x, P^1_2 x, P^1_3 x, \neg(P^1_1 x \wedge P^1_2 x \wedge P^1_3 x)\}$: unsatisfiable.

Similarly, we can find unsatisfiable sets of formulas of any finite size k with $k \geq 2$.

What about infinite sets of formulas?

In the same way, if a set has any of the above finite collections of formulas.

Now, suppose Γ is an infinite set of formulas s.t. all finite subsets of Γ are satisfiable. What happens then?

Compactness Theorem of F.O.L.

Let Γ be an infinite set of formulas. Then Γ is satisfiable iff Γ is finitely satisfiable (every finite subset of Γ is satisfiable)

Non-trivial part: If Γ is finitely satisfiable (fin-sat), then Γ is satisfiable (sat).

How does this result connect with the consequence relation?

① If Γ is fin-sat then Γ is sat.

② If $\Gamma \models \varphi$, then there is finite subset Γ_0 of Γ, s.t. $\Gamma_0 \models \varphi$.

Result : (1) iff (2) .

Proof : (2) $\Rightarrow$ (1) : Let Γ be fin. sat. To show that Γ is sat. Suppose not. Then, $\Gamma \models \varphi$ for all formulas φ. Then, there is a formula ψ, say s.t. $\Gamma \models \psi$ and $\Gamma \models \neg\psi$. So, there are :

$$\left.\begin{array}{l} - \Gamma_1 \subseteq_{fin} \Gamma \text{ s.t. } \Gamma_1 \models \psi \\ - \Gamma_2 \subseteq_{fin} \Gamma \text{ s.t. } \Gamma_2 \models \neg\psi \end{array}\right\} \text{ by (2)}$$

$\Gamma_1 \cup \Gamma_2 \models \psi \wedge \neg\psi$. So, $\Gamma_1 \cup \Gamma_2 \subseteq_{fin} \Gamma$ and $\Gamma_1 \cup \Gamma_2$ is not satisfiable, a contradiction. Hence, the result.

(1) $\Rightarrow$ (2) : Let $\Gamma \models \varphi$. To show that there is $\Gamma_0 \subseteq_{fin} \Gamma$ s.t. $\Gamma_0 \models \varphi$. Suppose not. So, for all $\Gamma_0 \subseteq_{fin} \Gamma$, $\Gamma_0 \not\models \varphi$. So, for all $\Gamma_0 \subseteq_{fin} \Gamma$, $\Gamma_0 \cup \{\neg\varphi\}$ is sat. Then by (1) $\Gamma \cup \{\neg\varphi\}$ is sat. Then, $\Gamma \not\models \varphi$, a contradiction. Hence the result.

More applications

- Let Γ be a set of sentences having arbitrarily large finite models. Then Γ has an infinite model.

<u>Proof</u>. Let $D = \{d_1, d_2, \ldots\}$ be a countable collection of new constant symbols not occurring in Γ. Consider

$$\Delta = \Gamma \cup \{\neg(d_i = d_j) \mid i, j \in \mathbb{N}, i \neq j\}$$

Now, Γ is satisfiable. So, Γ is finitely satisfiable. Take any finite subset of $\{\neg(d_i = d_j) \mid i, j \in \mathbb{N}, i \neq j\}$.

Such a finite set will be satisfiable in a model of Γ having that many distinct elements. So, we have that Δ is finitely satisfiable.

So, by compactness theorem Δ is satisfiable. But, a model of Δ has infinitely many elements.

Now, $\Gamma \subseteq \Delta$. So, a model of Δ is also a model of Γ.

Thus Γ has an infinite model.

This completes the proof. $\square$

Definability.

A class of first-order structures $\mathcal{K}$, say, is said to be first-order definable if there is a set of sentences Γ s.t. $\mathrm{Mod}(\Gamma) = \mathcal{K}$.

H.W. Let FIN denote the class of all finite structures. Show that FIN is not first-order definable.