Lecture 5

We proved the following result:

Let L be a fo-language. Let $\mathcal{A}$ and $\mathcal{B}$ be two L-structures. Let $h: D_{\mathcal{A}} \to D_{\mathcal{B}}$ be a map s.t. h is an isomorphism between the structures $\mathcal{A}$ and $\mathcal{B}$. Let $g_{\mathcal{A}}$ be an assignment function in the structure $\mathcal{A}$, and $g_{\mathcal{B}} = h \circ g_{\mathcal{A}}$ be the corresponding assignment function in the structure $\mathcal{B}$. Then, for all fo-formulas φ, $(\mathcal{A}, g_{\mathcal{A}}) \models \varphi$ iff $(\mathcal{B}, g_{\mathcal{B}}) \models \varphi$.

Corollary: Isomorphic structures are elementarily equivalent.

What about the converse? Are elementarily equivalent structures isomorphic?

NO! (We would consider the structures $(\mathbb{R}, <)$ and $(\mathbb{Q}, <)$ for counter-example) We get to this later.

Let us get to another application of the result above.

Proposition: Let $\mathcal{L}$ be an fo-language and $\mathcal{A}$ be an $\mathcal{L}$-structure. Let R be an n-ary relation on $D_{\mathcal{A}}$ such that R is definable in $\mathcal{L}$. Let h be an automorphism on $\mathcal{A}$. Then: $(a_1, \ldots, a_n) \in R$ iff $(h(a_1), \ldots, h(a_n)) \in R$.

Proof: Let $\varphi(x_1, \ldots, x_n)$ be a formula that defines R in $\mathcal{A}$. Then we have:

$$(a_1, \ldots, a_n) \in R$$

iff $\mathcal{A}_{[x_1 \to a_1, \ldots, x_n \to a_n]} \models \varphi$

iff $\mathcal{A}_{[x_1 \to h(a_1), \ldots, x_n \to h(a_n)]} \models \varphi$ (by the result above)

iff $(h(a_1), \ldots, h(a_n)) \in R$

This completes the proof. $\blacksquare$

H.W. Using this result show that $\{b\}$ is not definable in the example on graphs given earlier.

Another example

Consider the structure $(\mathbb{R}, <)$. Now, $\mathbb{N} \subseteq \mathbb{R}$. We show that $\mathbb{N}$ is not definable in $\mathbb{R}$, given the language $\mathcal{L}$ having a binary relation symbol, p, say, whose interpretation in $\mathbb{R}$ is given by $<$: We show this using by using an automorphism on $\mathbb{R}$: $h(x) = x^3$.

Since h is an automorphism, had $\mathbb{N}$ been definable, we should have:

$$n \in \mathbb{N} \text{ iff } h(n) \in \mathbb{N}$$

But, there are elements outside $\mathbb{N}$, which get mapped in $\mathbb{N}$. Thus $\mathbb{N}$ is not definable in $(\mathbb{R}, <)$. $\square$

Till now we have focussed on fo-language and expressivity. We will come back to these concepts, but for now, let us dive into the other important aspect of any logic, that is, reasoning in a given language. To this end we introduce:

(Semantic) Consequence relation:

Γ : a set of formulas.

φ : a formula.

We say that φ is a semantic consequence of Γ (denoted by $\Gamma \models \varphi$) if for all models $\mathcal{M}$, $\underbrace{\mathcal{M} \models \gamma \text{ for all } \gamma \in \Gamma}_{\mathcal{M} \models \Gamma}$ imply $\mathcal{M} \models \varphi$.

Example

$\Gamma = \{ px \to qx, \; qx \to rx \}$

$\varphi := px \to rx$

To show: $\Gamma \models \varphi$.

We need to show: for all models $\mathcal{M}$, if $\mathcal{M} \models \Gamma$, then $\mathcal{M} \models \varphi$.

Take any model $\mathcal{M} : (D, I, g)$.

Suppose $\mathcal{M} \models \Gamma$, that is, $\mathcal{M} \models px \to qx$ and $\mathcal{M} \models qx \to rx$

To show: $\mathcal{M} \models px \to rx$.

Let $\mathcal{M} \models px$ To show: $\mathcal{M} \models rx$.

Since $\mathcal{M} \models px$, $\mathcal{M} \models qx$ (as $\mathcal{M} \models px \to qx$).

Then, $\mathcal{M} \models rx$ (as $\mathcal{M} \models qx \to rx$)

This completes the proof. $\boxtimes$

Now, suppose $\Gamma = \Phi$. Then, $\Gamma \models \varphi$ means $\Phi \models \varphi$ (we denote by $\models \varphi$), which basically says: 'φ is satisfied by all models'.

- A formula φ is said to be valid if for all models $\mathcal{M}$, $\mathcal{M} \models \varphi$.

- A formula φ is said to be satisfiable if there is a model $\mathcal{M}$, such that $\mathcal{M} \models \varphi$.

Examples:

Consider an fo-language $\mathcal{L}$ with $\mathcal{C} = \Phi$, $\mathcal{F} = \Phi$, $\mathcal{P} = \{p^2\}$. Consider structures (D, I_k), where

D is a non-empty set. We can have various interpretations of p^2

$I_1(p^2) = D \times D \xleftrightarrow[\text{formula}]{\text{true}} \varphi_1 := \forall x \forall y\, p^2 xy$.

$I_2(p^2) = \phi \xleftrightarrow[\text{formula}]{\text{true}} \varphi_2 := \forall x \forall y \neg p^2 xy$.

$I_3(p^2) =$ a serial relation $\xleftrightarrow[\text{formula}]{\text{true}} \varphi_3 := \forall x \exists y\, p^2 xy$.

(for each $d \in D$, there is a $d' \in D$ s.t. $(d, d') \in I_3(p^2)$.)

Examples of valid and satisfiable formulas.

- $\forall x \forall y (p^2 xy \vee \neg p^2 xy)$ valid.
- $\forall x \forall y (p^2 xy \wedge \neg p^2 xy)$ true in only empty models
- $\exists x \exists y (p^2 xy \wedge \neg p^2 xy)$ unsatisfiable

M.W.

1. Let $\Gamma = \{\varphi, \neg\varphi\}$. Then show that $\Gamma \models \psi$ for all formulas ψ

2. Show that if $\varphi \in \Gamma$, then $\Gamma \models \varphi$.

3. Let φ be formula. Then, φ is valid iff $\neg\varphi$ is not satisfiable. Prove the statement.

4. Let $\Gamma_1 \subseteq \Gamma_2$ and $\Gamma_1 \models \varphi$. Then show that $\Gamma_2 \models \varphi$.

5. If $\Gamma \models \delta$ for $\delta \in \Delta$ and $\Delta \models \varphi$, then $\Gamma \models \varphi$.

6. $\Gamma \cup \{\varphi\} \models \psi$ iff $\Gamma \models \varphi \to \psi$.
Prove or disprove.

7. Check whether the following formulas are valid:

(a) $\forall x\,(px \to (qx \to px))$

(b) $(\exists x\, px \wedge \exists x\, qx) \to \exists x\,(px \wedge qx)$

(c) $\forall x\,(px \vee qx) \to (\forall x\, px \vee \forall x\, qx)$