LECTURE 6

31.01.2024

Proof of compactness theorem.

Statement: Let Γ be an infinite set of formulas. Then Γ is sat iff Γ is fin-sat.

Proof of only-if part: Suppose Γ is sat. To show that Γ is fin-sat. Since Γ is sat Γ has a model, $\mathcal{M}$, say. Then every finite subset of Γ would also have $\mathcal{M}$ as a model. So, Γ is fin-sat. This completes the proof.

Proof of the if-part: Suppose Γ is fin-sat. To show: Γ is sat. Since Γ is fin-sat, every finite subset of Γ has a model. To show that Γ is sat we have to find a model $\mathcal{M}$, say, of Γ, that is, we have to find $\mathcal{M} = (D, I, g)$ s.t. $\mathcal{M} \models \Gamma$.

Let us first ask the following question: What properties should Γ have for it to have a model?

To answer this question let us now look into a set of formulas which is satisfied by a model:

Let $\mathcal{M}$ be a model and, let

$$\Delta_{\mathcal{M}} = \{\varphi : \mathcal{M} \models \varphi\}.$$

What properties does $\Delta_{\mathcal{M}}$ have?

1. For any formula φ,

(a) Δ_μ contains φ or $\neg\varphi$.

(b) Δ_μ does not contain both.

2. For any formulas φ and ψ,

(a) $\varphi \vee \psi \in \Delta_\mu$ iff $\varphi \in \Delta_\mu$ or $\psi \in \Delta_\mu$

(b) $\varphi \wedge \psi \in \Delta_\mu$ iff $\varphi \in \Delta_\mu$ and $\psi \in \Delta_\mu$.

(c) $\varphi \to \psi \in \Delta_\mu$ iff $\psi \in \Delta_\mu$ whenever $\varphi \in \Delta_\mu$

(d) $\varphi \leftrightarrow \psi \in \Delta_\mu$ iff $\varphi \in \Delta_\mu$ iff $\psi \in \Delta_\mu$.

- A set of formulas satisfying conditions 1 and 2 is said to be a model set.

- A set of formulas Δ is said to be complete if for any formula φ, $\varphi \in \Delta$ or, $\neg\varphi \in \Delta$.

We will try to get a model of the fin-sat set Γ that we started with by proving the following 3 claims:

Claim 1: Every fin-sat set of formulas can be extended to a fin-sat and complete set of formulas.

Claim 2: Every fin-sat and complete set of formulas is a model set.

Claim 3: Every model set is satisfiable.

$$\underset{\text{fin-sat}}{\Gamma} \Longrightarrow \underset{\substack{\text{fin-sat} \\ + \\ \text{complete}}}{\Delta} \Longrightarrow \text{SAT}$$

Since $\Gamma \subseteq \Delta$, if Δ has a model, then Γ will also have a model. So, Γ is sat.

Proof of Claim 1:

We have that our language (set of all formulas) is countable. H.W.

Then, we can enumerate the set of formulas as follows:

$\varphi_0, \varphi_1, \varphi_2, - - - - -$

Let Γ be a fin-sat set of formulas. Now, we construct a sequence of sets of formulas as follows:

$$\Gamma_0 = \Gamma$$

$$\Gamma_{k+1} = \begin{cases} \Gamma_k \cup \{\varphi_k\} & \text{if } \Gamma_k \cup \{\varphi_k\} \text{ is fin-sat.} \\ \Gamma_k, & \text{otherwise} \end{cases}$$

So, $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq - - - -$

Take $\Delta = \bigcup_{n \geq 0} \Gamma_n$

We claim that Δ is fin-sat and complete.

- Δ is complete.

It is enough to show that for any set of formulas Γ' and any formula φ, if Γ' is fin-sat, then either $\Gamma' \cup \{\varphi\}$ is fin-sat or, $\Gamma' \cup \{\neg\varphi\}$ is fin-sat.

Proof: We assume Γ' is fin-sat. Also, suppose that neither $\Gamma' \cup \{\varphi\}$ nor, $\Gamma' \cup \{\neg\varphi\}$ is fin-sat. So, there exist $\Gamma_1 \subseteq_{fin} \Gamma' \cup \{\varphi\}$ and $\Gamma_2 \subseteq_{fin} \Gamma' \cup \{\neg\varphi\}$, s.t. Γ_1 and Γ_2 are not satisfiable. Then,

$\Gamma_1 = \Gamma_1' \cup \{\varphi\}$, $\Gamma_1' \subseteq_{fin} \Gamma$

$\Gamma_2 = \Gamma_2' \cup \{\neg\varphi\}$, $\Gamma_2' \subseteq_{fin} \Gamma$

Now, since $\Gamma_1' \cup \{\varphi\}$ is not sat, $\Gamma_1' \models \neg\varphi$

Also, since $\Gamma_2' \cup \{\neg\varphi\}$ is not sat, $\Gamma_2' \models \varphi$

So, $\Gamma_1' \cup \Gamma_2' \models \varphi \wedge \neg\varphi$.

Then, $\Gamma_1' \cup \Gamma_2'$ is not sat, a contradiction as, $\Gamma_1' \cup \Gamma_2' \subseteq_{fin} \Gamma'$. So, either $\Gamma' \cup \{\varphi\}$ is fin-sat or, $\Gamma' \cup \{\neg\varphi\}$ is fin-sat. This completes the proof.

- Δ is fin-sat.

Suppose Δ is not fin-sat. Then there is $\Delta' \subseteq_{fin} \Delta$ s.t. Δ' is not sat.

Now, $\Delta' \subseteq_{fin} \Gamma_k$ for some $k \geq 0$.

Then, Γ_k cannot be fin-sat, a contradiction to the construction of Γ_k. Hence, Δ is fin-sat.

This completes the proof of Claim 1. $\square$

Proof of Claim 2:

Let Δ be fin-sat and complete set of formulas. To show that Δ is a model set. Now, since Δ is fin-sat and complete, conditions 1(a) and 1(b) are satisfied. We now show conditions 2(a) - 2(d).

Condition 2(a): To show that $\varphi \vee \psi \in \Delta$ iff $\varphi \in \Delta$ or $\psi \in \Delta$.

- Suppose $\varphi \vee \psi \in \Delta$.

To show: $\varphi \in \Delta$ or $\psi \in \Delta$.

Suppose not. Then, $\varphi \notin \Delta$ and $\psi \notin \Delta$.

So, $\neg\varphi \in \Delta$ and $\neg\psi \in \Delta$

Then, $\{\varphi \vee \psi, \neg\varphi, \neg\psi\} \subseteq_{fin} \Delta$

But, $\{\varphi \vee \psi, \neg\varphi, \neg\psi\}$ is un-satisfiable

This contradicts the fact that Δ is fin-sat. Hence $\varphi \in \Delta$ or, $\psi \in \Delta$.

- Conversely, suppose $\varphi \in \Delta$ or $\psi \in \Delta$.

Without loss of generality, let $\varphi \in \Delta$.

To show $\varphi \vee \psi \in \Delta$.

Suppose not. Then, $\neg(\varphi \vee \psi) \in \Delta$.

Then, $\{\varphi, \neg(\varphi \vee \psi)\} \subseteq_{fin} \Delta$

But $\{\varphi, \neg(\varphi \vee \psi)\}$ is un-satisfiable.

This contradicts that Δ is fin-sat.

Hence $\varphi \vee \psi \in \Delta$.

This completes the proof of 2.(a).

H.W. Prove conditions 2(b) - 2(d).

This completes the proof of Claim 2.

Proof of Claim 3.

Let Δ be a model set. To show that Δ is sat, that is, to find a model $\mathcal{M}_\Delta$ s.t. $\mathcal{M}_\Delta \models \Delta$. Thus we need to find $\mathcal{M}_\Delta = (D_\Delta, I_\Delta, g_\Delta)$ s.t. $\mathcal{M}_\Delta \models \varphi$ iff $\varphi \in \Delta$.

Suppose we have $\mathcal{M}_\Delta$, somehow.

To prove $\mathcal{M}_\Delta \models \varphi$ iff $\varphi \in \Delta$.

We apply induction on the size of φ.

Base case: (i) φ : $t_1 \equiv t_2$

(ii) φ : $P_i^n\, t_1 t_2 \cdots t_n$

I.H. : Suppose the result holds for all formulas of size $\leq m$.

I.S.: Consider φ s.t. size of φ is $m+1$.

(1) $\varphi : \neg\psi$, (2) $\varphi : \psi \vee \chi$, $\psi \wedge \chi$, $\psi \to \chi$, $\psi \leftrightarrow \chi$.
(3) $\varphi : \forall x\, \psi$, $\exists x\, \psi$.

Suppose the result holds for base cases.

Consider Induction Step:

(1) $\varphi : \neg\psi$

$M_\Delta \models \varphi$ iff $M_\Delta \models \neg\psi$ iff $M_\Delta \not\models \psi$ iff $\psi \notin \Delta$ iff $\neg\psi \in \Delta$ iff $\varphi \in \Delta$

(2) $\varphi : \psi \vee \chi$.

$M_\Delta \models \varphi$ iff $M_\Delta \models \psi \vee \chi$ iff $M_\Delta \models \psi$ or $M_\Delta \models \chi$ iff $\psi \in \Delta$ or $\chi \in \Delta$ iff $\psi \vee \chi \in \Delta$ iff $\varphi \in \Delta$.

H.W. Prove the cases of $\psi \wedge \chi$, $\psi \to \chi$, $\psi \leftrightarrow \chi$

We now need to show the base cases and the induction step (Case 3).