Lecture 6

The use / abuse of the symbol $\models$

$\mathcal{M} \models \varphi$. $\qquad\qquad \Gamma \models \varphi$.

(satisfies) $\qquad\qquad$ (entails)

Models and Theories.

- Given any formula φ, define $Mod(\varphi) = \{\mathcal{M} : \mathcal{M} \models \varphi\}$. Given a set of formulas Γ, define $Mod(\Gamma) = \{\mathcal{M} : \mathcal{M} \models \Gamma\}$.

- Let $\mathcal{K}$ be a class of models. The theory of $\mathcal{K}$, denoted by $Th(\mathcal{K}) = \{\varphi : \mathcal{M} \models \varphi$ for all models $\mathcal{M}$ in $\mathcal{K}\}$.

1. Let Γ_1, Γ_2 be two sets of formulas s.t. $\Gamma_1 \subseteq \Gamma_2$. Then: $Mod(\Gamma_1) \supseteq Mod(\Gamma_2)$.

2. Let $\mathcal{K}_1, \mathcal{K}_2$ be two classes of models s.t. $\mathcal{K}_1 \subseteq \mathcal{K}_2$. Then: $Th(\mathcal{K}_1) \supseteq Th(\mathcal{K}_2)$.

Consequence of a set of formulas.

Let Γ be a set of formulas. Then consequence of Γ, denoted by $Con(\Gamma)$, is defined as $Con(\Gamma) = Th(Mod(\Gamma))$.

Proposition: Let Γ be a set of formulas and φ be a formula. Then: $\varphi \in Con(\Gamma)$ iff $\Gamma \models \varphi$.

H.W. Prove this proposition.

H.W. Prove the following properties of $Con(\Gamma)$.

1. $\Gamma \subseteq Con(\Gamma)$
2. if $\Gamma_1 \subseteq \Gamma_2$, then $Con(\Gamma_1) \subseteq Con(\Gamma_2)$.
3. $Con(Con(\Gamma)) = Con(\Gamma)$.

What does it mean to say that $\Gamma \not\models \varphi$?

$\Gamma \not\models \varphi$ iff there is a model $\mathcal{M}$, s.t. $\mathcal{M} \models \Gamma$ and $\mathcal{M} \not\models \varphi$.

iff there is a model $\mathcal{M}$, s.t $\mathcal{M} \models \Gamma$ and $\mathcal{M} \models \neg\varphi$.

iff there is a model $\mathcal{M}$, s.t. $\mathcal{M} \models \Gamma \cup \{\neg\varphi\}$.

iff $\Gamma \cup \{\neg\varphi\}$ is satisfiable.

More on satisfiability

Let Γ be a set of formulas and φ be a formula:

- φ is satisfiable if there is a model of φ
- Γ is satisfiable if there is a model of Γ

Example

Consider an fo-language $\mathcal{L}$ with relation symbols:

$p_1^1, p_2^1, p_3^1, \ldots\ldots$

1. $\Gamma = \{p_1^1 x\}$. Is Γ satisfiable?

To answer in the affirmative, we have to find a model (D, I, g) s.t. $(D, I, g) \models p_1^1 x$.

① $D = \{a, b\}$. $I(p_1^1) = \{a\}$, $g: V \to D$: $g(y) = a$ for all y.

Then, $(D, I, y) \models p_1^1 x$, as $y(x) \in I(p_1^1)$.

Thus, Γ is indeed satisfiable.

2. $\Gamma = \{p_1^1 x, \neg p_1^1 x\}$. Is Γ satisfiable?

Γ is unsatisfiable.

3. $\Gamma = \{p_1^1 x, p_2^1 x, \neg(p_1^1 x \wedge p_2^1 x)\}$

Γ is unsatisfiable

4. $\Gamma = \{p_1^1 x, p_2^1 x, p_3^1 x, \neg(p_1^1 x \wedge p_2^1 x \wedge p_3^1 x)\}$

Γ is unsatisfiable.

Similarly, we can find unsatisfiable sets of formulas of any finite size k with $k \geq 2$.

What about infinite sets of formulas?

In the same way, if a set has any of the above finite collections of formulas.

Now, suppose Γ is an infinite set of formulas s.t. all finite subsets of Γ are satisfiable.

What happens in this case?

Compactness Theorem of F.O.L.

Let Γ be an infinite set of formulas. Then, Γ is satisfiable iff every finite subset of Γ is satisfiable.

Non-trivial part: If Γ is finitely satisfiable (fin-sat), then Γ is satisfiable (sat).

How does this result connect with the consequence relation?

① If Γ is fin-sat, then Γ is sat.

② If $\Gamma \models \varphi$, then there is a finite subset Γ_f of Γ, s.t. $\Gamma_f \models \varphi$.

Result: ① iff ②

Proof: ② $\Rightarrow$ ①: Let Γ be fin-sat. To show that Γ is sat. Suppose not. Then, $\Gamma \models \varphi$ for all formulas φ. Then, there is a formula ψ, say, s.t. $\Gamma \models \psi$ and $\Gamma \models \neg\psi$. So, there are:

$\left.\begin{array}{l} - \Gamma_1 \subseteq_{fin} \Gamma \text{ s.t. } \Gamma_1 \models \psi \\ - \Gamma_2 \subseteq_{fin} \Gamma \text{ s.t. } \Gamma_2 \models \neg\psi \end{array}\right\}$ by 2

So, $\Gamma_1 \cup \Gamma_2 \models \psi \wedge \neg\psi$. Thus, $\Gamma_1 \cup \Gamma_2 \subseteq_{fin} \Gamma$ and $\Gamma_1 \cup \Gamma_2$ is not satisfiable; a contradiction. Hence, the result.

1 $\Rightarrow$ 2: Let $\Gamma \models \varphi$. To show that there is $\Gamma_f \subseteq_{fin} \Gamma$ s.t. $\Gamma_f \models \varphi$. Suppose not. So, for all $\Gamma_f \subseteq_{fin} \Gamma$, $\Gamma_f \not\models \varphi$. So, for all $\Gamma_f \subseteq_{fin} \Gamma$, $\Gamma_f \cup \{\neg\varphi\}$ is sat. Then, by 1 $\Gamma \cup \{\neg\varphi\}$ is sat. Then, $\Gamma \not\models \varphi$, a contradiction. Hence, the result. This completes the proof. $\blacksquare$

More applications of Compactness Theorem.

1. Let Σ be a set of sentences having arbitrarily large finite models. Then, Σ has an infinite model.

Proof: Let $\mathcal{D} = \{d_1, d_2, \cdots\}$ be a countable collection of new constant symbols not occuring in Σ_1. Consider $\Delta = \Sigma_1 \cup \{\neg(d_i = d_j) \mid i,j \in \mathbb{N}, i \neq j\}$.

Now, Σ_1 is satisfiable. So, Σ_1 is finitely satisfiable. Take any finite subset of $\{\neg(d_i = d_j) \mid i,j \in \mathbb{N}, i \neq j\}$. Such a finite subset will be satisfiable in some model of Σ_1. So, we have that Δ is finitely satisfiable. So, by compactness theorem, Δ is satisfiable. But, a model of Δ will contain infinitely many elements. Now, $\Sigma_1 \subseteq \Delta$. So, any model of Δ is also a model of Σ_1. Thus, Σ_1 has an infinte model. This completes the proof. ∎