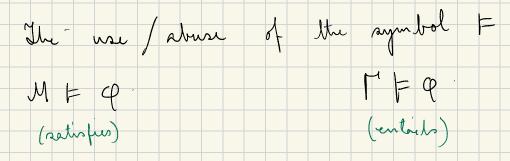
Lecture 6



Models and Theories

- Given any formula q, define Mod(q) = {M: M=q}. Given a set of formulas Γ', define Mod (M) = {M: MEM}.

- Let  $f_k$  be a class of models. The theory of  $f_k$ , denoted by  $T_k(f_k) = \{\varphi : M \neq \varphi\}$ for all models M in Id 3.

1. Let T, Tz be two sets of forum las  $\Gamma_1 \subseteq \Gamma_2$ . Then: Mod  $(\Gamma_1) \supseteq Mod (\Gamma_2)$ pt.

2. Let  $K_1$ ,  $K_2$  be Two classes of models s.t.  $K_1 \subseteq K_2$ . Then: Th $(K_1) \supseteq$  Tw $(K_2)$ . Consequence of a set of formulas het M be a set of for annels. Then consequence of M, denoted by Con (M), is defined as Con (r) = Th (Mod (r)) Proposition: Let M be a set of formulas and q be a formula. Then : q E Con (M) H Fq. H.W. Prove this proposition. H.W. Prove the following properties of (on (r) (1)  $\Gamma \subseteq Con(\Gamma)$ (2) if  $\Gamma'_1 \subseteq \Gamma'_2$ , then  $Con(\Gamma'_1) \subseteq Con(\Gamma'_2)$  $\operatorname{Con}\left(\operatorname{Con}\left(\Gamma\right)\right)=\operatorname{Con}\left(\Gamma\right).$ 3

does it mean to say that MH q! What iff there is a model M, s.t. MFT and M#P. r H 9 if there is a model M, s.t MET and ME19 iff there is a model M, s.t. MFMU{193 iff TU {1q] is satisfiable. Hore on satisfiability het M be a set of formulas and q be a formula q is satisfiable if there is a model of q is patisfiable if there is a model of T Example Consider an fo-language & with relation symbols P, , P2, P3, 1. T = { p, x }. Is T satisfiable? Jo answer in the affir mative, we have to find a model (D, I, M) n.t. (D, I, M) F b, x.  $D = \{a, b\}, I(b') = \{a\}, Y: V \rightarrow D: Y(y) = a \ fn \ all$ 

Then,  $(D, I, Y) \neq p'x$ , as  $Y(x) \in I(p', )$ Thus, M is inded satisfiable. 2.  $\Gamma = \{ b', x, \gamma b', \gamma \}$  Is  $\Gamma$  patingiable  $\gamma$ Mis unsatisfiable 3.  $M = \{ b, x, b, x, \tau (b, x \land b, x) \}$ M' is un satisfiable  $\begin{array}{c} 1 \\ 4 \\ \end{array}$ M is un sati spiable Simi landy, we can find unsatisfiable sets of formulas of any finite size k with k > 2. What about impinite sets of formules? In the same way, if a set has any of the above finite collections of formulas Now, suppose M is an infinite set of formulas s.t. all finete subsets of M are satisfiable.

What happens in this case ? Compactness Theorem of F.O.L. het M be an infinite set of formules. Then, T is satisfiable if every finite subset of M is satisfiable. Non-trivial part: If M is finitely satisfi-able (fin-sat), then M is satisfiable (sat) How does this result connect with The con sequence relation? 1) If T' is fin-sat, then T is sat 2) If MEQ, then there is a finite subset  $f_{f} of \Gamma, p.t. f_{f} \neq Q$ Result: Diff 2 Proof: 2 => 1 : Let I' be fin-sat. To show that I' is sat. Suppose not. Then, I' # q for all formulas q. Then, there is a formula y,

Ray, s.t. MEY and MEZY. So, there are:  $-\Gamma_{1} \subseteq_{fin} \Gamma \quad p.t. \quad \Gamma_{1} \neq \psi \qquad \\ -\Gamma_{2} \subseteq_{fin} \Gamma \quad p.t. \quad \Gamma_{2} \neq \gamma \psi \qquad \\ S \qquad by \textcircled{2}$ So, M,  $VF_2$  F  $\gamma$   $\Lambda$   $\gamma$   $\gamma$  Jhus, F,  $VF_2 \subseteq_{fin} F$ and  $\Gamma_1 \cup \Gamma_2$  is not patisfiable, a contra-diction. Hence, the result  $(1 \Rightarrow 2)$ . All  $\Gamma \neq \varphi$ . To show that there is  $\Gamma_f \subseteq_{fin} \Gamma$  p.t.  $\Gamma_f \not\models \mathcal{Q}$ . Suppose not . So, for all  $\Gamma_{f} \subseteq f_{in} \Gamma$ ,  $\Gamma_{f} \notin \varphi$ . So, for all I' fin I', I' U{1 p} is sat. Then, by 1) MU Erq3 is pat. Then, MHP, a contradiction. Hence, the result. This completes the proof. More applications of Compactness Theorem 1. Let Z, be a set of sentences training arbitrarily large finite model. Then, Z has an infinite model.

not occuring in Z. . Consider A  $= \sum_{i} \cup \{ \gamma(d_i = d_j) \mid i, j \in \mathbb{N}, i \neq j \}.$ Now, Z' is satisfiable. So, Z is finitely satisfiable. Take any finite publicit of  $\{7(di = dj) \mid i, j \in \mathbb{N}, i \neq j \}$ Such a finite subset will be patisfiable in some model of Z. So, we have that D is finitely satisfiable So, by compactness theorem, A is satisfiable But, a model of & will contain infinitely many elements. Now,  $\Sigma \subseteq \Delta$ . So, any model of  $\Delta$  is also a model of Z. Thus, Zi has an impirite model. This completes the proof. 12