Lecture 7

We already know that a class K of structures is definable in f.o. language if there is a sentence σ, s.t. $K = \{S : S \models \sigma\}$. Now, let us consider a weaker notion of definability:

A class K of structures is said be definable in f.o.l in the weaker sense if there is a set of sentences, Σ_1, say, s.t. $K = \{S : S \models \Sigma_1\}$, or, in other words, $K = Mod(\Sigma_1)$.

2. Proposition. Let FIN denote the class of all finite structures (structures with finite domain). FIN is not first-order definable, not even in the weaker sense.

H.W. Prove this proposition.

3. Downward Löwenheim-Skolem Theorem:

Let Γ be a satisfiable set of formulas. Then Γ has a countable model.

Proof idea: Since Γ is satisfiable, Γ is finitely satisfiable. Using this fin-sat assume, we can come up with a countable model of Γ through a proof of compactness theorem.

Important Fact: First-order language is countable, in other words, the set of formulas is countable.

Consider the two questions posed earlier where we have not yet given a formal answer:

1. Is the class of infinite sets definable in F.O.L?
2. How to show that elementary equivalence

does not imply isomorphism ?

Answer to 1.

The answer is NO. Had the class of all infinite sets, $\mathcal{I}$, say, be definable, that is, $\mathcal{I} = \mathrm{Mod}(\tau)$, for some first-order sentence τ, we have that the class of finite structures could be written as: $\mathrm{FIN} = \mathrm{Mod}(\neg\tau)$. This is not possible by the Application (2.) above. ▨

Hence the claim.

Answer to 2.

Consider an fo-language $\mathcal{L}_<$ say with a single parameter $<$, a two-place relation symbol. Consider $(\mathbb{R}, <_{\mathbb{R}})$ and $(\mathbb{Q}, <_{\mathbb{Q}})$, two $\mathcal{L}_<$-structures. Of course, they are not isomorphic.

In the following we will show that they are elementarily equivalent.

To prove the above statement we will use the following result (without proving it):

- Any two countable linear dense order without end-points are isomorphic (Cantor)

(A good exercise to try out!)

Theories

- A first-order theory is a set of sentences T s.t. for all sentences σ, if $T \models \sigma$ then $\sigma \in T$.

- In other words, a set T of sentences is a theory iff $T = \mathrm{Con}(T)$.

Complete theories

A theory T is said to be complete if for every sentence σ, either $\sigma \in T$ or, $\neg\sigma \in T$.

Proposition: A theory T is complete iff any two models of T are elementarily equivalent.

H.W. Prove this proposition.

$\aleph_0$-categorical theory:

A theory T is said to be $\aleph_0$-categorical if all its models of cardinality $\aleph_0$ are isomorphic.

Łos-Vaught Test: Let T be theory; If T is $\aleph_0$-categorical, then T is complete.

Proof: To show that T is complete, it

is enough to show that for any two models, $\mathcal{A}$ and $\mathcal{B}$, say, of T, $\mathcal{A} \equiv \mathcal{B}$.

For $\mathcal{A}$, consider $\Sigma_{\mathcal{A}} = \{\sigma : \mathcal{A} \models \sigma\}$.

Then $\Sigma_{\mathcal{A}}$ is a satisfiable set of sentences. Then, by Downward Löwenheim Skolem theorem, $\Sigma_{\mathcal{A}}$ has a countable model, $\mathcal{A}'$, say.

Then, $\mathcal{A} \equiv \mathcal{A}'$. Also, $\mathcal{A}'$ is a model of T, as $T \subseteq \Sigma_{\mathcal{A}}$. Similarly, we get a countable model $\mathcal{B}'$ of T s.t. $\mathcal{B} \equiv \mathcal{B}'$. So, $\mathcal{A}'$ and $\mathcal{B}'$ are countable models of T.

Now, suppose T is $\aleph_0$-categorical. Then, $\mathcal{A}'$ is isomorphic to $\mathcal{B}'$. So, $\mathcal{A}' \equiv \mathcal{B}'$.

Then, we have: $\mathcal{A} \equiv \mathcal{A}' \equiv \mathcal{B}' \equiv \mathcal{B}$. Since, $\mathcal{A}$ and $\mathcal{B}$ are any two models of T, we have that T is complete. This completes the proof. $\blacksquare$

We are now all set to show $(\mathbb{R}, <_{\mathbb{R}}) \equiv (\mathbb{Q}, <_{\mathbb{Q}})$

Let $L_<$ be the language of order : $<$.

Consider the following set S of sentences:

1. (a) $\forall x \forall y \, (x < y \vee x = y \vee y < x)$

   (b) $\forall x \forall y \, (x < y \to \neg (y < x))$

   (c) $\forall x \forall y \forall z \, (x < y \to (y < z \to x < z))$

2. $\forall x \forall y \, (x < y \to \exists z \, (x < z \wedge z < y))$

3. $\forall x \exists y \exists z \, (y < x \wedge x < z)$.

Let $T = Con(S)$ is a theory whose models are $(\mathbb{R}, <_{\mathbb{R}})$ and $(\mathbb{Q}, <_{\mathbb{Q}})$. Now, as T is the theory of dense linear orders without end-points, from Cantor's theorem we have that T is $\aleph_0$-categorical.

Then, Łoś-Vaught test tells us that T is complete. So, all models of T are elementarily equivalent. Thus, $(\mathbb{R}, <_{\mathbb{R}}) \equiv (\mathbb{Q}, <_{\mathbb{Q}})$. This completes the discussion.