Lecture 7

05.02.2024

Let us now move on to the base cases.

We first try to define M_Δ.

Define $D_\Delta = \{t : t \text{ is a term}\}$.

Define I_Δ:

- $I_\Delta(c) = c$ for all $c \in \mathcal{C}$.
- $I_\Delta(f_i^n) : D_\Delta^n \to D_\Delta$ given as follows:

$$I_\Delta(f_i^n)(t_1, t_2, \ldots, t_n) = f_i^n t_1 t_2 \cdots t_n$$

- $I_\Delta(p_i^n) \subseteq D_\Delta^n$ given as follows:

$$(t_1, t_2, \ldots, t_n) \in I_\Delta(p_i^n) \text{ iff } p_i^n t_1 t_2 \cdots t_n \in \Delta.$$

Define $\mathcal{G}_\Delta$:

$$\mathcal{G}_\Delta(x) = x \text{ for all } x \in V.$$

So, $M_\Delta = (D_\Delta, I_\Delta, \mathcal{G}_\Delta)$

Now, g_X can be extended uniquely to $g'_\Delta : \mathcal{T} \to D_\Delta$, and hence, we also mention g'_Δ as g_Δ.

H.W. $g_\Delta(t) = t$ for all terms $t \in \mathcal{T}$.

Now, we show that $\mathcal{M}_\Delta \models \varphi$ iff $\varphi \in \Delta$.

Base cases: (1) $\varphi : t_1 \equiv t_2$
(2) $\varphi : p_i^n t_1 t_2 \cdots t_n$

Let us prove (2) first:

$\mathcal{M}_\Delta \models \varphi$
iff $\mathcal{M}_\Delta \models p_i^n t_1 t_2 \cdots t_n$
iff $(g_\Delta(t_1), g_\Delta(t_2), \cdots, g_\Delta(t_n)) \in I_\Delta(p_i^n)$
iff $(t_1, t_2, \cdots, t_n) \in I_\Delta(p_i^n)$
iff $p_i^n t_1 t_2 \cdots t_n \in \Delta$
iff $\varphi \in \Delta$. This complete the proof.

Now we get into the base case (1)

To prove: $\mathcal{M}_\Delta \models t_1 \equiv t_2$ iff $t_1 \equiv t_2 \in \Delta$.

Now, $\mathcal{M}_\Delta \models t_1 \equiv t_2$ iff $y_\delta(t_1) = y_\delta(t_2)$

iff $t_1 = t_2$ **iff** $t_1 \equiv t_2 \in \Delta$.

$t_1 = t_2 = t$

(in the language of arithmetic:

1 = 1 , 2 = 2 , ... ,

but, we would also like to equate 2 + 1 and 3)

Let us go back to the domain definition. We had $D_\Delta = \{t : t \text{ is a term}\}$

Let us define a relation $\approx$ on $\mathcal{T}$ as follows: $t_1 \approx t_2$ iff $t_1 \equiv t_2 \in \Delta$.

- $\approx$ is an equivalence relation.

(i) $\approx$ is reflexive.

We have to show that $t \equiv t \in \Delta$ for all terms t. Suppose not. There is a term t, s.t. $t \equiv t \notin \Delta$.

Since, Δ is a model set, Δ is complete and so, $\neg(t \equiv t) \in \Delta$.
So, $\{\neg(t \equiv t)\}$ is satisfiable, a contradiction. So, $t \equiv t \in \Delta$ for all terms t. Hence, $\approx$ is reflexive.

H.W. (2) $\approx$ is symmetric.
(3) $\approx$ is transitive.

This completes the proof that $\approx$ is an equivalence relation. □

This equivalence relation gives rise to a collection of equivalence classes over $\mathcal{T}$, i.e., D_Δ.

Define $D'_\Delta = \{[t] : t \text{ is a term}\}$;
where $[t]$ denotes the equivalence class of t under the equivalence relation defined above.
Next, we would define I'_Δ and g'_Δ.