LECTURE 8

07.02.2024

We have:

- $I'_\Delta(c) = [c]$
- $I'_\Delta(f^n_i) : (D'_\Delta)^n \to D'_\Delta$ defined as follows:

$$I'_\Delta(f^n_i)([t_1],[t_2],\ldots,[t_n]) = [f^n_i t_1 t_2 \ldots t_n]$$

- $I'_\Delta(p^n_i) \subseteq (D'_\Delta)^n$ defined as follows:

$$([t_1],[t_2],\ldots,[t_n]) \in I'_\Delta(p^n_i) \text{ iff } p^n_i t_1 t_2 \ldots t_n \in \Delta.$$

Here, we need to check that $I'_\Delta(f^n_i)$ and $I'_\Delta(p^n_i)$ are well-defined.

- $I'_\Delta(f^n_i)$ is well-defined.

Let $t_1, t_2, \ldots, t_n, t'_1, t'_2, \ldots, t'_n$ be terms s.t. $t_1 \approx t'_1, t_2 \approx t'_2, \ldots, t_n \approx t'_n$. To show: $f^n_i t_1 t_2 \ldots t_n \approx f^n_i t'_1 t'_2 \ldots t'_n$.

That is we have to show:

If $t_1 \equiv t_1', t_2 \equiv t_2', \ldots, t_n \equiv t_n' \in \Delta$, then
$$f_i^n t_1 t_2 \cdots t_n \equiv f_i^n t_1' t_2' \cdots t_n' \in \Delta.$$

Proof: Suppose $t_1 \equiv t_1', t_2 \equiv t_2', \ldots, t_n \equiv t_n' \in \Delta$.

[Lemma: Let φ be a formula. Then $\models \varphi$ implies $\varphi \in \Delta$ (fin-sat and complete)

Proof: Suppose $\models \varphi$ but $\varphi \notin \Delta$. Then, $\neg \varphi \in \Delta$. So $\{\neg \varphi\}$ is satisfiable, a contradiction. Hence $\varphi \in \Delta$]

Now we have:
$$\models (t_1 \equiv t_1' \wedge t_2 \equiv t_2' \wedge \cdots \wedge t_n \equiv t_n') \rightarrow (f_i^n t_1 t_2 \cdots t_n \equiv f_i^n t_1' t_2' \cdots t_n')$$

H.W. Prove this validity.

So, by the lemma we just proved,
$$(t_1 \equiv t_1' \wedge t_2 \equiv t_2' \wedge \cdots \wedge t_n \equiv t_n') \rightarrow (f_i^n t_1 t_2 \cdots t_n \equiv f_i^n t_1' t_2' \cdots t_n') \in \Delta.$$

Now, as $t_1 \equiv t_1', t_2 \equiv t_2', \ldots, t_n \equiv t_n' \in \Delta$, $t_1 \equiv t_1' \wedge t_2 \equiv t_2' \wedge \cdots \wedge t_n \equiv t_n' \in \Delta$, Δ being a model set. So, we have:

$f_i^n t_1 t_2 \cdots t_n \equiv f_i^n t_1' t_2' \cdots t_n' \in \Delta$.

So, $f_i^n t_1 t_2 \cdots t_n \approx f_i^n t_1' t_2' \cdots t_n'$, whenever, $t_1 \approx t_1'$, $t_2 \approx t_2'$, $\cdots$, $t_n \approx t_n'$. This completes the proof of well-definedness of $I_\Delta'(f_i^n)$. □

H.W. $I_\Delta'(p_i^n)$ is well-defined.

Now, let us define g_Δ' as follows:

$g_\Delta'(x) = [x]$, for all variables $x \in V$.

So, we have our model:

$$\boxed{M_\Delta' = (D_\Delta', I_\Delta', g_\Delta')}$$

As earlier, we can show that $g_\Delta'(t) = [t]$ for all terms $t \in J$. H.W.

Let us go back to the result we have to prove:

For all formulas φ, $M_\Delta' \models \varphi$ iff $\varphi \in \Delta$.

Base Case: (1) $\varphi : t_1 \equiv t_2$: $\mathcal{M}'_\Delta \models \varphi$ iff $\mathcal{M}'_\Delta \models t_1 \equiv t_2$ iff $g'_\Delta(t_1) = g'_\Delta(t_2)$ iff $[t_1] = [t_2]$ iff $t_1 \approx t_2$ iff $t_1 \equiv t_2 \in \Delta$ iff $\varphi \in \Delta$.

(2) $\varphi : p^n_i t_1 t_2 \cdots t_n$: $\mathcal{M}'_\Delta \models \varphi$ iff $\mathcal{M}'_\Delta \models p^n_i t_1 t_2 \cdots t_n$ iff $(g'_\Delta(t_1), g'_\Delta(t_2), \cdots, g'_\Delta(t_n)) \in I'_\Delta(p^n_i)$ iff $([t_1], [t_2], \cdots, [t_n]) \in I'_\Delta(p^n_i)$ iff $p^n_i t_1 t_2 \cdots t_n \in \Delta$ iff $\varphi \in \Delta$.

So, we are done for the base cases.
Now, by our previous argument we are also done for the formulas $\neg\psi$, $\psi \vee \chi$, $\psi \wedge \chi$, $\psi \to \chi$, $\psi \leftrightarrow \chi$. Thus if consider the atomic formulas (base cases) and their boolean combinations (done earlier), then we have the compactness theorem for the

logical language ;

$$\varphi, \psi ::= t_1 \equiv t_2 \mid p_i^n t_1 t_2 \cdots t_n \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid$$
$$\varphi \to \psi \mid \varphi \leftrightarrow \psi .$$

We call this language <u>zeroth order language</u> for zeroth order logic.

<u>The quantifier case</u>

Let us now consider the quantified formulas $\forall x \psi$ and $\exists x \psi'$. We will be done if we can show that :

$\mathcal{M}'_\Delta \models \forall x \psi$ iff $\forall x \psi \in \Delta$

$\mathcal{M}'_\Delta \models \exists x \psi'$ iff $\exists x \psi' \in \Delta$.

<u>How do we proceed ?</u>

Now, $\mathcal{M}'_\Delta \models \forall x \psi$ iff for all $d \in D'_\Delta$, $\mathcal{M}_{\Delta[x \mapsto d]} \models \psi$.

Here, $D'_\Delta = \{[t] : t \text{ is a term}\}$.

To make sense out of this we have to talk about <u>terms replacing variables in formulas</u>. We introduce the concept of substitutions.

<u>Substitutions</u>

- in : Terms

Let t and t' be two terms and x be a variable. We write $t[t'/x]$ for t' replacing x in the term t.

- $x[t'/x] = t'$
- $y[t'/x] = y$, $y \neq x$
- $c[t'/x] = c$
- $f_i^n t_1 t_2 \cdots t_n [t'/x] = f_i^n\, t_1[t'/x]\, t_2[t'/x] \cdots t_n[t'/x]$

- in : Formulas

Let φ be a formula, t be a term an x be a variable. Then, we write $\varphi[t/x]$ for x replaced by t in φ.

- $(t_1 \equiv t_2)[t/x] = (t_1[t/x] \equiv t_2[t/x])$.

- $p_i^n t_1 \cdots t_n [t/x] = p_i^n t_1[t/x] \cdots t_n[t/x]$

- $\neg\psi[t/x] = \neg(\psi[t/x])$

- $(\psi \vee \chi)[t/x] = \psi[t/x] \vee \chi[t/x]$

- $(\psi \wedge \chi)[t/x] = \psi[t/x] \wedge \chi[t/x]$

- $(\psi \to \chi)[t/x] = \psi[t/x] \to \chi[t/x]$

- $(\psi \leftrightarrow \chi)[t/x] = \psi[t/x] \to \chi[t/x]$

- $\begin{cases} \forall x \psi[t/x] = \forall x \psi . \\ \forall y \psi[t/x] = \forall y(\psi[t/x]), \quad y \neq x \end{cases}$

- $\begin{cases} \exists x \psi[t/x] = \exists x \psi \\ \exists y \psi[t/x] = \exists y(\psi[t/x]), \quad y \neq x \end{cases}$

Examples

1. $(x \equiv y)[y/x] = y \equiv y$

2. $(x \equiv y)[x/y] = x \equiv x$.

3. $\forall x (x \equiv y)[y/x] = \forall x (x \equiv y)$

4. $\forall x (x \equiv y)[x/y] = \forall x (x \equiv x)$

Now, we would like to consider satisfiability of such subsituted formulas.

Let $\mathcal{M}$ be a model and consider a substituted formula $\varphi[t/x]$. What would be a natural way to think about $\mathcal{M} \models \varphi[t/x]$, where $\mathcal{M} = (D, I, g)$?

$\mathcal{M} \models \varphi[t/x]$ iff $\mathcal{M}_{[x \to g(t)]} \models \varphi$.

Would this always hold? NO

φ : $\exists y(\neg(y = x))$: sat

$\varphi[y/x]$: $\exists y(\neg(y = y))$: unsat

Substitutability

A term t is said to be substitutable for a variable x in a formula φ when the following hold:

- φ is atomic : t is always substitutable for x in φ.
- φ is $\neg\psi$: t is substitutable for x in ψ.
- φ is $\psi \vee \chi$: t is substitutable for x in ψ and t is substitutable for x in χ.

- φ is $\psi \wedge \chi$: "

- φ is $\psi \rightarrow \chi$ "

- φ is $\psi \leftrightarrow \chi$: "

- φ is $\forall y \psi$: x does not occur free in ψ.

OR

y should not appear in the term t and t is substitutable for x in ψ.

- φ is $\exists y \psi$: "

Proposition : Let φ be a formula, x be a variable, t be a term, $\mathcal{M}$ be a model. Then, $\mathcal{M} \models \varphi[t/x]$ iff $\mathcal{M}_{[x \to \mathcal{I}(t)]} \models \varphi$, provided t is substitutable for x in φ.

H.W. Prove this proposition.