LECTURE 9

12.02.2024

Our target is to show the following:

$$\boxed{\mathcal{M}'_\Delta \models \forall x \varphi \text{ iff } \forall x \varphi \in \Delta}.$$

Now we know that $\models \forall x \varphi \leftrightarrow \neg \exists x \neg \varphi$.

With this we would prove one side of the above 'iff' result using the properties of the formula $\forall x \varphi$ and for the other side, we would use the properties of $\exists x \varphi$.

How would we go about that?

We would try to show the following:

① if $\forall x \varphi \in \Delta$, then $\varphi[t/x] \in \Delta$ for all terms t.

② if $\exists x \varphi \in \Delta$, then $\varphi[t/x] \in \Delta$ for some term t.

For ①, let us first prove the following:

Proposition: $\models \forall x \varphi \to \varphi[t/x]$, where, t is substitutable for x in φ.

Proof: Let $\mathcal{M}$ be a model s.t. $\mathcal{M} \models \forall x \varphi$. Then, for any d in $D_{\mathcal{M}}$, $\mathcal{M}_{[x \to d]} \models \varphi$. So, in particular, we have $\mathcal{M}_{[x \to g_{\mathcal{M}}(t)]} \models \varphi$, where t is substitutable for x in φ. Then, we have: $\mathcal{M} \models \varphi[t/x]$. Thus $\mathcal{M} \models \forall x \varphi$ implies $\mathcal{M} \models \varphi[t/x]$ for any model $\mathcal{M}$, where t is substitutable for x in φ. Hence, $\models \forall x \varphi \to \varphi[t/x]$, where t is substitutable for x in φ. This completes the proof. $\square$

So, we get ① for all those terms t which are substitutable for x in φ.

Let us now move on to ②. To get the condition ②, let us introduce the following concept of witness-fulfilled sets of formulas:

Witness-fulfilled.

A set of formulas Δ is said to be witness-fulfilled if for every formula of the form $\exists x \varphi$, if $\exists x \varphi \in \Delta$, then $\varphi[t/x] \in \Delta$ for some term t.

An example :

Consider the language of arithmetic. Let P be any unary predicate symbol. Consider the set

$$\Delta = \{\exists x P(x), \neg P(0)\} \cup \{\neg P(S^k(0)) : k \geq 1\}$$

This is an infinite set of formulas which is fin-sat. This should be satisfiable by 'compactness theorem'. And, it actually is. To ensure such a property, we introduced the above concept of witness-fulfilledness.

Proposition: Any fin-sat, complete set of formulas Γ can be extended to a fin-sat complete and witness-fulfilled set of formulas.

Proof: We note that a constant symbol c not occuring in a formula φ can always be substitutable for x in φ. We use this idea below.

We expand our language $\mathcal{L}$ with a countable set of new constant symbols, $\mathcal{D}$, say, where $\mathcal{D} = \{d_1, d_2, \ldots\}$. So the parameters of the new language $\mathcal{L}'$ is given by the parameters of $\mathcal{L}$, the language we started off together with the set $\mathcal{D}$. In other words, $\mathcal{L} : (\mathcal{C}, \mathcal{F}, \mathcal{P})$ and $\mathcal{L}' : (\mathcal{C} \cup \mathcal{D}, \mathcal{F}, \mathcal{P})$.

As $\mathcal{L}$ is countable, $\mathcal{L}'$ is also countable

Let us enumerate the formulas in $\mathcal{L}'$ as follows: $\beta_0, \beta_1, \beta_2, \ldots\ldots$

We now construct sets of formulas $\Delta_0, \Delta_1, \Delta_2, \ldots$ as follows:

- $\Delta_0 = \Gamma$ (fin-sat and complete).
- If $\beta_k = \exists x \varphi$, then,

$$\Delta_{k+1} = \begin{cases} \Delta_k \cup \{\exists x \varphi, \varphi[d/x]\}, & \text{if it is fin-sat, where } d \text{ is the least-indexed constant symbol, not occurring in } \Delta_k \\ \Delta_k, & \text{otherwise} \end{cases}$$

- else,

$$\Delta_{k+1} = \begin{cases} \Delta_k \cup \{\beta_k\}, & \text{if it is fin-sat} \\ \Delta_k, & \text{otherwise.} \end{cases}$$

We have: $\Delta_0 \subseteq \Delta_1 \subseteq \Delta_2 \subseteq \ldots$

Let $\Delta = \bigcup_{n \geq 0} \Delta_n$ ...

Claim: Δ is fin-sat, complete, witness-fulfilled.