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Second Order Logic

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In the first place, one should ask that *"Why even bother about second-order logic while we were doing pretty fine with our first-order logic, which we already learnt, and are familiar with?"*. Well, as most of the new ideas and concepts were born, so did the second-order logic, which arose from some drawbacks of the first-order logic, which can not talk about things like *"for all properties"*. In the philosophy of mathematics the second-order logic is often the centre of some heated arguments among scholars. It is stronger than first order logic in that it incorporates *"for all properties"* into the syntax, while first order logic can only say *"for all elements"*. At the same time it is arguably weaker than set theory in that its quantifiers range over one limited domain at a time, while set theory has the universalist approach in that its quantifiers range over all possible domains. This stronger-than-first-order-logic/weaker-than-set-theory duality is the source of lively debate, not least because set theory is usually construed as based on first order logic. How can second-order logic be at the same time stronger and weaker? To make things worse, it was suggested that a first order set-theoretic background has to be assumed to make use of the strength of second order logic to full extent, and give an exact interpretation to the *"for all properties"*. This not only undermines the claimed strength of second order logic as well as its role as the primary foundation of mathematics, but also fails to bypass the set theoretic aspects that the second order logic would have wanted to, namely - *the higher infinite, the independence results*, and *the difficulties in finding new convincing axioms*. We still use set theory as the metatheory for modern mathematics, although second order logic aimed to be a better substitute for it (That would make mathematics complicated, since set theory is much more well developed). Setting aside philosophical questions, it is undeniable and manifested by a continued stream of interesting results, that second-order logic is part and parcel of a logician's toolbox, especially in computer science logic and finite model theory. Central questions of theoretical computer science, such as the $P = NP$? question, can be seen as questions about second-order logic in the finite context.

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1 Introduction

Consider elementary number theory. The objects of our study are the natural numbers 0, 1, 2,... and their arithmetic. With first order logic we can formulate statements about number theory by using atomic expressions $x = y, x+y = z$ and $x*y = z$ combined with the propositional operations $\wedge, \vee, \neg, \rightarrow, \longleftrightarrow$ and the quantifiers $\forall x$ and $\exists x$. Here the variables x, y, z,... are thought to range over the natural numbers. With second-order logic our scope of variables grow larger: in addition to the already existing interpretation for variables, we have variables X, Y, Z,... for properties of numbers and relations between numbers as well as quantifiers $\forall X$ and $\exists X$ for these variables. We have the atomic expressions of first order logic and also new atomic expressions of the form $X(y_1, \ldots, y_n)$.

One may ask, *"Which properties of natural numbers can be expressed/proved in/by second order logic but not first order logic?"*.

With just the constant 0 and the unary function $n \mapsto n^+$ (where n^+ means n+1) we can express in second-order logic the *Induction Axiom* of natural numbers:

$$\forall X([X(0) \wedge \forall y(X(y) \rightarrow X(y^+))] \rightarrow \forall y X(y)) \tag{1}$$

This, together with the axioms $\forall x \neg(x^+ = 0)$ and $\forall x \forall y(x^+ = y^+ \rightarrow x = y)$ characterizes the successor operation of natural numbers (up to isomorphism).

In first order logic any theory which has a countably infinite model has also an uncountable model (by *Upward Löwenheim Skolem Theorem*). Hence (1) cannot be expressed in first order logic.

Another second-order expression is the *Completeness Axiom* of the linear order $\leq$ of the real numbers:

$$\forall X([\exists y X(y) \wedge \exists z \forall y(X(y) \rightarrow y \leq z)] \rightarrow \exists z \forall y(\exists u(X(u) \wedge y \leq u) \vee z \leq y)) \tag{2}$$

This, together with the axioms of ordered fields characterizes the ordered field of real numbers (up to isomorphism). In first order logic any countable theory which has an infinite model has also a countable model (by *Downward Löwenheim Skolem Theorem*). Hence (2) cannot be expressed in first order logic.

In early days, logicians including *Russell, Löwenheim, Hilbert* and *Zermelo* didn't think second order logic was much different form first order logic. In 1929 *Gödel* proved his *Completeness Theorem* and the next year his *Incompleteness Theorem*. This, along with later developments emphasized that second order logic is much different. *Gödel* showed that any effective axiomatization of number theory is incomplete. On the other hand, there was a simple finite categorical (hence complete) axiomatization of the structure $(N, +, *)$ in second-order logic. This showed that there cannot be such a complete axiomatization of second-order logic as there was for first order logic.

Later, *Henkin* proved the *Completeness Theorem* for second-order logic, allowing us to think of both first and second order logic as the same way, just keeping in mind that the semantics is based on general models.

2 The Syntax of Second-Order Logic

A vocabulary in second-order logic is similar to a vocabulary in first order logic. Remember that first order logic vocabulary consists of :

A Logical symbols

- i Parentheses: (,)
- ii Sentential connective symbols: $\rightarrow, \neg, \longleftrightarrow$.
- iii Variables (one for each positive integer n): $v_1, v_2, ...$
- iv Equality symbol (optional): $=$

B Parameters

- i *Quantifier symbol*: $\forall, \exists$
- ii *Predicate symbols*: For each positive integer n, some set (possibly empty) of symbols, called n-place predicate symbols
- iii *Constant symbols*: Some set (possibly empty) of symbols
- iv *Function symbols*: For each positive integer n, some set (possibly empty) of symbols, called n-place function symbols.

In second order logic, we have some more logical symbols.

- *Predicate variables (Also called Property or Relation variables)*: For each positive integer n, we have the n-place predicate variables $X_1^n, X_2^n, ...$
- *Function variables*: For each positive integer n, we have the n-place function variables $F_1^n, F_2^n, ...$

The usual variables $v_1, v_2, ...$ will now be called *individual variables*, to avoid confusion.

It is noteworthy that although we have property variables we do not have variables for properties of properties. Such variables would be part of the formalism of third order logic.

Definition ***Terms:*** *The terms are as before defined as the expressions that can be built up from the constant symbols and the individual variables by applying the function symbols (both the function parameters and the function variables)*

- *Constant symbols and individual variables are terms*
- *If $t_1, t_2, ..., t_n$ are terms, U is an n-ary function symbol and F is an n-ary function variable, then $U(t_1, t_2, ..., t_n)$ and $F(t_1, t_2, ..., t_n)$ are terms. Note that terms denote individuals, not relations or properties. Thus X alone is not a term but x is.*

Definition ***Atomic Formulas:*** *Atomic formulas are defined from terms as follows:*

- *If t and t' are terms, then $t = t'$ is an atomic formula*
- *If R is an n-ary relation symbol (parameter or variable) and $t_1, t_2, ..., t_n$ are terms, then $R(t_1, t_2, ..., t_n)$ is an atomic formula*

- *If X is an n-ary relation variable, then also $X(t_1, t_2, ..., t_n)$ is an atomic formula.*

Definition ***Formulas:*** *The formulas of second-order logic are defined as follows -*

- *Atomic formulas are formulas*
- *If ϕ and ψ are formulas, then $\neg\phi, \phi \wedge \psi, \phi \vee \psi, \phi \rightarrow \psi$ and $\phi \longleftrightarrow \psi$ are formulas*
- *If ϕ is a formula, x an individual variable, X a relation variable and F a function variable, then $\exists x\phi, \forall x\phi, \exists X\phi, \forall X\phi, \exists F\phi$ and $\forall F\phi$ are formulas*

It is interesting to note that in second order logic we can actually define the identity $t = t'$ as $\forall X(X(t) \longleftrightarrow X(t'))$ and prove the familiar axioms of identity from properties of the implication.

An important special case is *monadic second-order logic* where no function variables are allowed and the relation variables are required to be *monadic* (a.k.a. unary), i.e., of arity one.

Remark 1

i *We did not take $X = Y$ as an atomic formula (although we could have) but having introduced the quantifiers we can use*

$$\forall x_1 \forall x_2 ..... \forall x_n (X(x_1, x_2, ..., x_n) \longleftrightarrow Y(x_1, x_2, ..., x_n))$$

as a substitute for $X = Y$. This gives the identity $X = Y$ an extensional flavour in contrast to a possibly different intensional construal

ii *The concepts of a free and bound occurrence of a variable in a formula are defined in the usual way (as done in first order logic).*

iii *A formula σ is called a sentence if it has no free variable.*

3 The Semantics of Second-Order Logic

Here we discuss about a set-theoretical interpretation of second-order logic, interpreting "properties" as sets.

Structure:

Formally, a *structure* $\mathfrak{M}$ for our given second-order language is a function whose domain is the set of parameters and such that :

1. $\mathfrak{M}$ assigns to the quantifier symbol $\forall$ a nonempty set $|\mathfrak{M}|$ called the universe (or domain) of $\mathfrak{M}$
2. $\mathfrak{M}$ assigns to each n-place predicate symbol P an n-ary relation $P^{\mathfrak{M}} \subseteq |\mathfrak{M}|^n$; i.e., $P^{\mathfrak{M}}$ is a set of n-tuples of members of the universe
3. $\mathfrak{M}$ assigns to each constant symbol c a member $c^{\mathfrak{M}}$ of the universe $|\mathfrak{M}|$
4. $\mathfrak{M}$ assigns to each n-place function symbol f an n-ary operation $f^{\mathfrak{M}}$ on $|\mathfrak{M}|$; i.e., $f^{\mathfrak{M}} : |\mathfrak{M}|^n \rightarrow |\mathfrak{M}|$

5. $\models_{\mathfrak{M}} \forall X_i^n \phi[s]$ iff for every n-ary relation R on $|\mathfrak{M}|$, we have $\models_{\mathfrak{M}} \phi[s(X_i^n|R)]$

6. $\models_{\mathfrak{M}} \forall F_i^n \phi[s]$ iff for every n-ary function $f : |\mathfrak{M}|^n \to |\mathfrak{M}|$, we have $\models_{\mathfrak{A}} \phi[s(F_i^n|f)]$

Assignment Function:

Given an L-structure $\mathfrak{M}$, an assignment is a function s from variables to the domain M of $\mathfrak{M}$ such that:

- if x is an individual variable, then $s(x) \in M$
- if X is relation variable of arity n, then $s(X) \subseteq M^n$
- if F is function variable of arity n, then $s(F) : M^n \to M$

We use $s(P/X)$ to denote the assignment which is otherwise as s except that the value at X has been changed to P. Similarly $s(a/x)$ and $s(f/F)$.

Note *1. 0-ary relation variables are essentially propositions. Their interpretations under an assignment are the truth values (true, false). 0-ary function variables are essentially individual variables as an assignment maps a 0-ary function symbol simply to an element of M*

2. The value $t^{\mathfrak{M}}<s>$ of a term t in a model $\mathfrak{M}$ under the interpretation s is defined as in first order logic.

3. It is easy to see that only the values of s at variables occurring free in the formula are significant. For a sentence σ, we may unambiguously speak of it being true or false in $\mathfrak{M}$. Logical (semantical) implication is defined exactly as before, while truth definition is given below -

Definition ***Tarski's Truth Definition:*** *The truth definition for second-order logic extends the respective truth definition for first order logic by the clauses :*

1. $\mathfrak{M} \models_s X(t_1, t_2, ..., t_n)$ *iff* $(t_1^{\mathfrak{M}} < s >, t_2^{\mathfrak{M}} < s >, ..., t_n^{\mathfrak{M}} < s >) \in s(X)$

2. $\mathfrak{M} \models_s \exists X \phi$ *iff* $\mathfrak{M} \models_{s(P/X)} \phi$ *for some* $P \subseteq M^n$

3. $\mathfrak{M} \models_s \exists F \phi$ *iff* $\mathfrak{M} \models_{s(f/F)} \phi$ *for some* $f : M^n \to M$

and similarly for the universal quantifiers. For a sentence ϕ, we define $\mathfrak{M} \models \phi$ to mean $\mathfrak{M} \models_s \phi$ for all (equivalently some) s, and then say ϕ is true in $\mathfrak{M}$

Validity and Equivalence of Formulas:

We say that ϕ is (logically) *valid* if $\mathfrak{M} \models_s \phi$ holds for all $\mathfrak{M}$ and all s. Likewise, we define ϕ and ψ to be *logically equivalent* (i.e., $\phi \equiv \psi$), if $\phi \longleftrightarrow \psi$ is valid. Two models $\mathfrak{M}$ and $\mathfrak{N}$ are said to be second-order equivalent (in symbols, $\mathfrak{M} \equiv_{L^2} \mathfrak{N}$ if for all sentences ϕ we have $\mathfrak{M} \models \phi \iff \mathfrak{N} \models \psi$.

For infinite M the collection of subsets of M^n and the set of functions $M^n \to M$ are famously complex. To reflect the difficulties involved with finding a $P \subseteq M^n$ or an $f : M^n \to M$ we sometimes say we "guess" a $P \subseteq M^n$ or an $f : M^n \to M$.

3.1 The *Ehrenfeucht-Fraïssé* game of Second-Order Logic

The main idea behind the game is that we have two structures, and two players – Spoiler (player I) and Duplicator (player II). Duplicator wants to show that the two structures are elementarily equivalent (satisfy the same first-order sentences), whereas Spoiler wants to show that they are different. The game is played in rounds. A round proceeds as follows: Spoiler chooses any element from one of the structures, and Duplicator chooses an element from the other structure. In simplified terms, the Duplicator's task is to always pick an element "similar" to the one that the Spoiler has chosen, whereas the Spoiler's task is to choose an element for which no "similar" element exists in the other structure. Duplicator wins if there exists an isomorphism between the eventual substructures chosen from the two different structures; otherwise, Spoiler wins.

The game lasts for a fixed number of steps n (which is an ordinal – usually a finite number.

For simplicity we disallow function and constant symbols as well as function variables in this section. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are two models of the same finite relational vocabulary.

The Game:

The game is denoted by $G^2_n(\mathfrak{A}, \mathfrak{B})$. 2 players I and II pick subsets (or elements) of A and B one at a time. The steps are as following -

- **Round 1** : player I can pick a relation A_i on A (or an element a_i of A) and then player II has to pick a relation B_i on B of the same arity as A_i (or an element b_i of B) and vice versa. player I can instead pick a relation B_i on B (or an element b_i of B) and in that case player II has to pick a relation A_i on A of the same arity as B_i (or an element a_i of A).
- **Next Rounds** : Subsequent rounds go the same way till n-th step.
- **Decider** : After n rounds, the pairs of chosen elements (a_i, b_i) form a binary relation R on A×B. If this relation is a partial isomorphism of the structures $\mathfrak{A}$ and $\mathfrak{B}$ expanded by the played relations A_i and B_i, i.e., it preserves atomic formulas and their negations, we say that Duplicator has won.

Strategy:

Unfortunately, unlike the first order case, the game $G^2_n(\mathfrak{A}, \mathfrak{B})$ is much more complex and the Spoiler is always at an advantage. Duplicator, in generality, has no winning strategy except the trivial case where $\mathfrak{A} \equiv \mathfrak{B}$. However, if we restrict to *monadic second-order logic*, which in terms of the *Ehrenfeucht-Fraïssé* game means restricting the game to unary predicates, the situation changes. A unary predicate just divides the model into two parts. If Player I divides A into two parts, Player II should find a similar division in B. This is more reasonable and there actually are useful strategies for Player II.

4 Some Examples

Let's see some concrete examples of second order logic structures to feel that we are in control -

Example 1 *A well-ordering is an ordering relation such that any nonempty set has a least (with respect to the ordering) element. This last condition can be translated into the second-order sentence :*

$$\forall X(\exists\ y\ Xy \rightarrow \exists y(Xy \wedge \forall z(Xz \rightarrow y \leq z)))$$

Example 2 *One of Peano's postulates (the induction postulate) states that any set of natural numbers that contains 0 and is closed under the successor function is, in fact, the set of all natural numbers. This can be translated into the second-order language for number theory as*

$$\forall X(X0 \wedge \forall y(Xy \rightarrow XSy) \rightarrow \forall y Xy)$$

Where S is the successor operator. Any model that satisfies

$$\forall x Sx \neq 0 \qquad \textit{and}$$

$$\forall X \forall y(Sx = Sy \rightarrow x = y)$$

and the above Peano induction postulate is isomorphic to $\Re_S = (\mathbb{N}; 0, S)$. Thus this set of sentences is categorical; i.e., all its models are isomorphic.

Example 3 *For any formula ϕ in which the predicate variable X_n does not occur free, the formula*

$$\exists X_n \forall v_1 \forall v_2 ..... \forall v_n [X^n v_1 v_2 ..... v_n \longleftrightarrow \phi]$$

is valid. (Here other variables may occur free in ϕ in addition to $v_1, v_2, ..., v_n$.) It says that there exists a relation consisting of exactly the n-tuples satisfying ϕ. Formulas of this form are called relation comprehension formulas. There are also the analogous function comprehension formulas. If ψ is a formula in which the variable F_n does not occur free, then

$$\forall v_1 \forall v_2 ..... \forall v_n \exists! v_{n+1} \psi \rightarrow \exists F_n \forall v_1 \forall v_2 ..... \forall v_{n+1} (F^n v_1 v_2 ..... v_n = v_n + 1 \longleftrightarrow \psi)$$

is valid.

Example 4 *In the ordered field of real numbers, any bounded nonempty set has a least upper bound. We can translate this by the second-order sentence -*

$$\forall X[\exists y \forall z(Xz \rightarrow z \leq y) \wedge \exists z Xz \rightarrow \exists y \forall y'(\forall z(Xz \rightarrow z \leq y') \longleftrightarrow y \leq y')]$$

It is known that any ordered field that satisfies this second-order sentence is isomorphic to the ordered field of real numbers $\mathbb{R}$.

Example 5 *For each $n \geq 2$, we have a first-order sentence λ_n which translates, "There are at least n things." For example, λ_3 is*

$$\exists x \exists y \exists z(x \neq y \wedge x \neq z \wedge y \neq z)$$

The set $\{\lambda_2, \lambda_3, ...\}$ has for its class of models the $EC_\triangle$ class consisting of the infinite structures. There is a single second order sentence that is equivalent. A set is infinite iff *there is an ordering on it having no last element. Or more simply, a set is infinite iff there is a transitive irreflexive relation R on the set whose domain is the entire set. This condition can be translated into a second-order sentence*

$$\lambda_\infty : \exists X[\forall u \forall v \forall w(Xuv \to Xvw \to Xuw) \land \forall u \neg Xuu \land \forall u \exists v Xuv].$$

Another sentence (using a function variable) that defines the class of infinite structures is

$$\exists F[\forall x \forall y(Fx = Fy \to x = y) \land \exists z \forall x Fx \neq z]$$

which says there is a one-to-one function that is not onto.

5 The Infamous Power of Second-Order Logic

5.1 The Collapse of Compactness Theorem

The *Compactness Theorem* is one of the cornerstones of our understanding of first order logic. We shall now see that there is **no hope of a *Compactness Theorem* for second-order logic!** (although that can be done by modifying the semantics as in *Henkin* models or general model setup)

Theorem 1 *There is an unsatisfiable set of second-order sentences every finite subset of which is satisfiable.*

Proof. We have already seen the counter-example in example 5 of section 4. The desired set of infinite formulas is, in the notation of that example, $\{\neg\lambda_\infty, \lambda_2, \lambda_3, ...\}$ □

The *Löwenheim Skolem Theorem* also fails for second-order logic! By the language of equality we mean the language (with =) having no parameters other than ∀. A structure for this language can be viewed as being simply a nonempty set. In particular, a structure is determined to within isomorphism by its cardinality. A sentence in this language is therefore determined to within logical equivalence by the set of cardinalities of its models (called its *spectrum*).

Theorem 2 *There is a sentence in the second-order language of equality that is true in a set iff its cardinality is* $2^{\aleph_0}$ *.*

Proof. (Using concepts from Algebra and Analysis) Consider first the conjunction of the (first-order) axioms for an ordered field, further conjoined with the second-order sentence expressing the least-upper-bound property (see Example 4 of section 4). This is a sentence whose models are exactly the isomorphs of the real ordered field (i.e., the structures isomorphic to the ordered field of real numbers). We now convert the parameters 0, 1, +, ·, < to variables (individual, function, or predicate as appropriate) which we existentially quantify. The resulting sentence has the desired properties. □

We now state a theorem about undefinability of validity in second order logic. But we will not delve into the proof since that requires a bit more understanding of first order logic and its analogues in second order case.

Theorem 3 ***(Hintikka 1955; Montague 1963)*** *: Validity in second-order logic is not second-order definable over* $(\mathbb{N},+,.,0)$.

Theorem 4 ***(Alternate Statement of the above)*** *: The set of Gödel numbers of valid second-order sentences is not definable in* $\mathbb{N}$ *by any second-order formula.*

Note *A fortiori, the set of Gödel numbers of second-order validities is not arithmetical and not recursively enumerable. That is, the enumerability theorem fails for second-order logic. (In the other direction, one can show that this set is not definable in number theory of order three, or even of order* ω*. But these are topics we will not cover here.*

5.2 "*Set Theory in Sheep's Clothing*"

"Second-order logic hides in its semantics some of the most difficult problems of set theory" - Resnik

In Philosophy of Logic [*Quine, 1970*], the author summed up a popular opinion among mathematical logicians by referring to second-order logic as *"set theory in sheep's clothing"*.

Let us see where this opinion might come from. First we observe that a very basic, second-order formula can say that two sets have the same cardinality: Suppose P and R are unary relation variables. Let $\theta_{\leq}(P,R)$ be the formula

$$\exists F(\forall x\forall y((F(x)=F(y)\rightarrow x=y)\wedge(P(x)\rightarrow R(F(x))))$$

Now $\mathfrak{M}\models_s \theta_{\leq}(P,R) \iff |s(P)|\leq|s(R)|$. Let $\phi(P,R)$ be the formula $\theta_{\leq}(P,R)\wedge\theta_{\leq}(R,P)$. Then $\mathfrak{M}\models_s \phi(P,R) \iff |s(P)|=|s(R)|$. Let $\theta'_{EC}(Y)$ be

$$F(\forall x\forall y((F(x)=F(y)\rightarrow x=y)\wedge R(F(x)))\wedge\forall x\exists y(R(x)\rightarrow x=F(y)))$$

Now $\mathfrak{M}\models_s \phi(P,R) \iff$ the sets M and s(R) have the same cardinality. We will use these formulas to launch an attack on the *Continuum Hypothesis* or *C.H.*(*there is no set whose cardinality is strictly between that of the integers and the real numbers, or equivalently, that any subset of the real numbers is finite, is countably infinite, or has the same cardinality as the real numbers.*

Let θ_{CH} be the sentence -

$$\exists E\exists U\exists G\exists z(\theta_{Pow}(E,U)\wedge\theta_{PA}(U,G,z)\wedge\forall Y(\theta'_{EC}(Y)\vee\theta\leq(Y,U)))$$

Now θ_{CH}, which is a sentence of the empty vocabulary, has a model if and only if the *C.H.* holds. Similarly, there is a sentence $\theta_{\neg CH}$, which has a model if and only if the *C.H.* does not hold. This shows that the dependence of the semantics of second-order logic on the metatheoretic set theory is so deep that even questions that ZFC cannot solve can determine the truth or falsity of a sentence in a model.

6 Model Theory of Second-Order Logic

6.1 Second-Order Characterizable Structures

A structure $\mathfrak{A}$ is second-order characterizable if there is a second-order sentence $\theta_{\mathfrak{A}}$ such that $\mathfrak{B}\models\theta_{\mathfrak{A}} \iff \mathfrak{B}\equiv\mathfrak{A}$ for all structures $\mathfrak{B}$ of the same vocabulary as $\mathfrak{A}$.

Example 6 *The following structures are second-order characterizable:*

1. *Natural numbers:* $(\mathbb{N}, +, .)$
2. *Real numbers:* $(\mathbb{R}, +, ., 0, 1)$
3. *Complex numbers:* $(\mathbb{C}, +, ., 0, 1, i)$
4. *The first uncountable ordinal* $(\omega_1, <)$
5. *The level* (V_κ, ϵ) *of the cumulative hierarchy, where* κ *is the first strongly inaccessible cardinal* $> \omega$
6. *The well-order* $(\kappa, <)$ *of the first weakly compact cardinal* $> \omega$

Are all structures second-order characterizable? There are only countably many second-order sentences, hence only countably many (up to isomorphism) second-order characterizable structures. Therefore there are lots of structures of every infinite cardinality which are not second-order characterizable. However, it is not easy to give examples. One example is $(\kappa, <)$, where κ is the first measurable cardinal $(> \omega)$. Another example is $(\mathbb{N}, <, A)$, where A is the set of *Gödel* numbers of valid second-order sentences in the vocabulary of one binary relation.

A special property of second-order characterizable structures is that their reducts are also second-order characterizable, because we can use existential second-order quantifiers to "guess" the missing relations and functions. Therefore it is interesting to find characterizable structures that have as many (but only finitely many) relations and functions as possible. We can endow $\mathbb{N}$ with any finite number of recursive functions $f_1, ..., f_n$ and relations $R_1, ..., R_m$ obtaining the structure $(\mathbb{N}, f_1, ..., f_n, R_1, ..., R_m)$ and this is second-order characterizable. We can endow $\mathbb{R}$ with any familiar analytic functions such as trigonometric functions or any other functions given by a convergent power series the coefficients of which are given by a recursive function, and the result is second-order characterizable.

The bigger the characterizable structure is, the more complex is the second-order theory. That there is no largest characterizable structure can be seen as follows: If $\mathfrak{A}$ is second-order characterizable, then so is the reduct of $\mathfrak{A}$ to the empty vocabulary, that is, the cardinality $|\mathfrak{A}|$ of $\mathfrak{A}$ is characterizable. Such cardinal numbers were studied by *Garland,(1974)*. For example, if κ is characterizable, then so are κ^+ and 2^κ.

If ϕ is a second-order sentence, we define

$$Mod(\phi) = \{\mathfrak{M} : \mathfrak{M} \models \phi\}$$

If ϕ characterizes a model $\mathfrak{A}$, then Mod(ϕ) is just the class of models isomorphic to $\mathfrak{A}$.

7 Second Order Arithmetic

The second-order theory of natural numbers, known as second-order arithmetic and denoted by Z_2, is an important foundational theory. It is stronger than (first order) *Peano* arithmetic but weaker than set theory. It has variables for individuals thought of as natural numbers as well as variables for sets of natural numbers thought of as real numbers. In addition there are $+$ and $\times$ for arithmetic operations on the individuals. As axioms Z_2 has some rather obvious axioms about $+$ and $\times$, the Induction Axiom (1), and the axioms of second-order logic (this little scope of discussion is not enough to explore them), including the Comprehension Principle -

$$\exists R \forall x_1 .....x_n(\phi(x_1, , ..., x_n) \longleftrightarrow R(x_1, ..., x_n))$$

where $\phi(x_1, ..., x_n)$ is a second-order formula with $x_1, ..., x_n$ among its free individual variables and the second-order variable R is not free in ϕ.

A surprising amount of mathematics can be derived in Z_2. In a sense, Z_2 is a great success story for second-order logic.

Reverse mathematics uses Z_2 to isolate the exact axioms on which well-known theorems from mathematics rely. In a textbook such theorems are proved perhaps in an informal set theory, but how much set theory is actually needed in each case? For example, we may ask what is the weakest set of axioms from which the *Bolzano-Weierstrass Theorem* can be proved? How much set theory, comprehension, choice, induction, etc is needed? Since Z_2 is a natural and sufficient environment for many mathematical theorems, it is an appropriate framework for answering questions raised by the reverse mathematics program. The main (but not the only) distinctions that are made in reverse mathematics concern the amount of the *Comprehension Principle* that is needed in proving this or that mathematical result.

8 Second Order Set Theory

We have up to now treated set theory (ZFC) as a first order theory. However, when *Zermelo,(1930)* introduced the axioms which constitute the modern ZFC axiom system, he formulated the axioms in second-order logic. In particular, his *Separation Axiom* is

$$\forall x \forall X \exists y \forall z(z \in Y \longleftrightarrow (z \in x \wedge X(z)))$$

and the *Replacement Axiom* is

$$\forall x \forall F \exists y \forall z(z \in y \longleftrightarrow \exists u(u \in x \wedge z = F(u)))$$

Second-order ZFC, ZFC^2, is simply the received first order ZFC with the Separation Schema replaced by the above single Separation Axiom, and the Replacement Schema replaced by the above single Replacement Axiom. Accordingly, ZFC^2 is a finite axiom system. *Zermelo* proved that the models of ZFC^2 are, up to isomorphism, of the form $(V_\kappa, \in)$, where κ is (strongly) inaccessible ($> \omega$).

Second-order set theory in a sense decides the C.H., i.e., decides whether it is true or not, even if we do not know which way the decision goes. More exactly $ZFC^2 \models$ C.H. or $ZFC^2 \models \neg$ C.H., because C.H. is true if and only if $V_\kappa \models$ C.H. for inaccessible κ, i.e., if and only if $ZFC^2 \models$ C.H.. Of course we can express C.H. in first order set theory, too, so the situation is not really different from first order set theory. Many set theorists think that the concept of set is definitive enough to decide eventually also C.H. even if ZFC does not decide it. Likewise, we may argue that the concept of second-order semantics is definitive enough to decide C.H. even if the current axioms of second-order logic cannot do it.

One may ask, *why do we not use second-order set theory ZFC^2 as the metatheory of second-order logic?*. In fact we could use it. However, the question might rise, what is the semantics of our metatheory? In principle such questions can lead to an infinite regress. By using first order set theory as the metatheory, the question about the semantics of the metatheory would simply be the question about the semantics of first order logic. Note that semantics of first order logic is absolute relative to ZFC. This gives some assurance that we need not continue asking, what the metatheory is.

9 Miscellaneous

This little scope of discussion hardly gives us enough flavour of the mathematical monstrosity that the second order logic is. But still, it gives us a nice overview of what it is, how things are corresponding to this language, and why it is important.

Bibliography

The following books and online archives have been really helpful for me -

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Other Internet Resources

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