Second Order Logic

Logic for Computer Science Project

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Why study Second order logic?

- First order language can not express every structure. One drawback is that it talks about the elements and axiomatizes their properties, but it can not talk about the properties themselves.
- For example - graphs, completeness of linear orders are not F.O.L expressible.
- Let us take number theory as an example. With first order logic we can express properties such as “n is a prime number” and propositions such as “there are infinitely many prime numbers”, Fermat’s Last Theorem and Goldbach’s Conjecture.
- But it can not express the properties of sets of natural numbers or properties of real numbers or define rational numbers.

Duality of Second Order Logic

- First Order Logic talks about “for all elements” whereas Second Order Logic can talk about “for all properties” also
- Hence one can say that it is stronger than its first order counterpart
- But quantifiers of second order logic cover only one limited domain at a time, while set theory has universalist approach in that regard
- This suggests second order logic is weaker than set theory, which is construed as based on first order logic - which is a contradictory statement!
- Early logicians aimed at creating second order logic as a better substitute for set theory for the role of a mathematical metatheory, but it didn’t go well - mainly because set theory is much simpler and well constructed
- Still second order logic finds its importance and utility in computer science logic and finite model theory

Introduction to Second order logic

- Frege first introduced us to the concept of Second-order logic in his book “Begriffsschrift”(1879) (German, roughly means ”concept-writing”). He also coined the term “second order” (in German, “zweiter Ordnung”) in (1884). It was widely used in logic until the 1930s, when set theory started to take over.
- In first order logic we had variables, predicate symbols, function symbols, logical connectives and quantifiers
- In second order language, the scope of variable grows larger - along with the variables we had from first order logic (call them individual variables), we also consider the predicate symbols and function symbols to be variables (called predicate/property/relation variable and function variable, respectively)
- The following is an example of a sentence of second order logic: With just the constant 0 and the unary function $n \rightarrow n+$ (where n+ means n+1) we can express in second-order logic the Induction Axiom of natural numbers:

$$\forall X([X(0) \land \forall y(X(y) \rightarrow X(y+))] \rightarrow \forall y X(y))$$

Introduction to Second order logic

- Here X is a unary relation variable, ie it talks about some property associated to each natural number.
- The above sentence, together with the axioms $\forall x \neg(x^+ = 0)$ and $\forall x \forall y(x^+ = y^+ \Rightarrow x = y)$ characterizes the successor operation of natural numbers (up to isomorphism)
- In first order logic any theory which has a countably infinite model has also an uncountable model (by Upward Lowenheim Skolem Theorem). But our model is Natural numbers (or some isomorphic copy of it), which is obviously countable. Hence that sentence cannot be expressed in first order logic.

Introduction to Second order logic

- Another example is the

Completeness Axiom of the linear order ≤ of the Real numbers:

$\forall X([\exists y X(y) \wedge \exists z \forall y(X(y) \to y \leq z)] \to \exists z \forall y(\exists u(X(u) \wedge y \leq u) \vee z \leq y))$

- This, together with the axioms of ordered fields characterizes the ordered field of real numbers (up to isomorphism).
- In first order logic any countable theory which has an infinite model has also a countable model (by Downward Lowenheim Skolem Theorem). Hence this also is not expressible in first order logic

Syntax of Second order logic

- We keep the vocabulary of first order logic in its entirety, and add some more. Remember that in first order logic, we had the following things -
- Logical symbols : parentheses, connective symbols (→, ¬, ←→), variables (which we call individual variables in second order logic), and equality sign (optional)
- Parameter symbols : quantifiers (∀, ∃), predicate symbols, function symbols, constant symbols (we allow the later three to be possibly empty sets)
- In second order logic, we include function variables and predicate variables
- ***Terms :*** The terms in first order logic maintain their property as terms. Additionally,If $t_1, t_2, ..., t_n$ are terms, U is an n-ary predicate symbol and F is an n-ary function variable, then $U(t_1, t_2, ..., t_n)$ and $F(t_1, t_2, ..., t_n)$ are terms
- Note that F or U alone are not terms but $F(t_1, t_2, ..., t_n)$ and $U(t_1, t_2, ..., t_n)$ are

Syntax and Semantics of Second order logic

- It is noteworthy that although we have property variables we do not have variables for properties of properties. Such variables would be part of the formalism of third order logic
- ***Atomic Formula :*** We stick to our previous definition of atomic formula (note that term now has more extensive meaning - hence that modifies the definition accordingly)
- ***Formula*** : Again we stick to our previous definition of formulas as in first order logic (just append the now larger scope of terms into the scenario everywhere)
- ***Free and bound occurrence of variables :*** As in first order logic
- ***Sentence :*** A formula with no free variable
- Similarly, model, structure, assignment functions, interpretation, truth definition, validity and equivalence of formulas are all concepts borrowed from those of first order logic

The Ehrenfeucht-Fraiss´e game of Second-Order Logic

- The aim of this game is to investigate to what point 2 models, say $\mathcal{A}$ and $\mathcal{B}$ are similar. There is one Spoiler and one Duplicator. The Spoiler wants to show they are different while duplicator aims for similarity.
- The game goes in rounds. Number of rounds is pre-decided. At each round Spoiler picks a relation or an element from either model while the Duplicator, in return, has to provide a relation (of same arity) or an element, accordingly, from the other model.
- At the conclusion of the game, we have chosen distinct elements $a_1, a_2, \ldots, a_t$ from $\mathcal{A}$ and $b_1, b_2, \ldots, b_t$ from $\mathcal{B}$ along with some relations from both of them. If these structures are partial isomorphic, Duplicator wins, otherwise Spoiler takes the game

The Ehrenfeucht-Fraıss´e game of Second-Order Logic

- Hence the strategy for Spoiler is that he always tries to pick element for which no "similar" element exists in the other structure
- On the other hand, the task for Duplicator is always pick an element "similar" to the one that the Spoiler has chosen
- The game in second order case is much complex compared to the first order case. Spoiler is always at advantage
- Duplicator has no guaranteed strategy to win unless the trivial case, when the models are isomorphic

Examples

- A well-ordering is an ordering relation such that any nonempty set has a least (with respect to the ordering) element. This last condition can be translated into the second order sentence :

$$\forall X(\exists\ y\ Xy \rightarrow \exists y(Xy \land \forall z(Xz \rightarrow y \leq z)))$$

- The induction postulate states that any set of natural numbers that contains 0 and is closed under the successor function is, in fact, the set of all natural numbers. This can be translated into the second-order language for number theory as

$$\forall X(X0 \land \forall y(Xy \rightarrow XSy) \rightarrow \forall yXy)$$

Where S is the successor operator. Any model that satisfies $\forall xSx \neq 0$ and $\forall X\forall y(Sx = Sy \rightarrow x = y)$ and the above Peano induction postulate is isomorphic to $(N; 0, S)$. Thus this set of sentences is categorical; i.e., all its models are isomorphic

Examples

- In the ordered field of real numbers, any bounded nonempty set has a least upper bound. We can translate this by the second-order sentence -

$$\forall X[\exists y\forall z(Xz \to z \leq y) \land \exists z Xz \to \exists y\forall y(\forall z(Xz \to z \leq y) \leftrightarrow y \leq y)]$$

It is known that any ordered field that satisfies this second-order sentence is isomorphic to the ordered field of real numbers $\mathcal{R}$.

- For each $n \geq 2$, we have a first-order sentence λ_n which translates, “There are at least n things.” For example, λ_3 is $\exists x\exists y\exists z(x \neq y \land x \neq z \land y \neq z)$. The set $\{\lambda_2, \lambda_3, \ldots\}$ has for its class of models the class consisting of the infinite structures. There is a single second order sentence that is equivalent. A set is infinite iff there is a transitive irreflexive relation R on the set whose domain is the entire set. This is given by

$$\lambda_\infty : \exists X[\forall u\forall v\forall w(Xuv \to Xvw \to Xuw) \land \forall u\neg Xuu \land \forall u\exists v Xuv]$$

Collapse of Compactness Theorem

- ***Theorem 1 :*** There is an unsatisfiable set of second-order sentences every finite subset of which is satisfiable.
- ***Proof :*** We have already seen the counter-example in the last example. The desired set of infinite formulas is, in the notation of that example, $\{\neg \lambda_{\infty}, \lambda_2, \lambda_3, ...\}$
- Note that the statement of this theorem violates the compactness theorem. Hence second order logic fails to have a compactness theorem
- The Lowenheim Skolem Theorem also fails for second-order logic!
- ***Theorem 2 :*** There is a sentence in the second-order language of equality that is true in a set iff its cardinality is $2^\wedge\aleph_0$

Second-Order Characterizable Structures

- The following are some examples of second-order characterizable structures :

1. Natural numbers: ($\mathcal{N}$, +, .)
2. Real numbers: ($\mathcal{R}$, +, ., 0, 1)
3. Complex numbers: ($\mathcal{C}$, +, ., 0, 1, i)

- There are only countably many second order sentences, hence only countably many (up to isomorphism) second-order characterizable structures. Therefore there are lots of structures of every infinite cardinality which are not second-order characterizable. However, examples are not trivial

One such example is ($\mathcal{N}$, <, A), where A is the set of Godel numbers of valid second-order sentences in the vocabulary of one binary relation.

Second Order Arithmetic

- The second-order theory of natural numbers, known as second-order arithmetic and denoted by Z_2, is an important foundational theory. It is stronger than (first order) Peano arithmetic but weaker than set theory.
- It has variables for individuals thought of as natural numbers as well as variables for sets of natural numbers thought of as real numbers
- Z_2 has some rather obvious axioms about + and ×, the Induction Axiom, and the axioms of second order logic, including the Comprehension Principle -

$$\exists R \forall x_1 \ldots\ldots x_n (\varphi(x_1, , \ldots\ldots, x_n) \leftrightarrow R(x_1, \ldots\ldots, x_n))$$

where $\varphi(x_1, \ldots\ldots, x_n)$ is a second-order formula with $x_1, \ldots\ldots, x_n$ among its free individual variables and the second-order variable R is not free in φ.

Second Order Arithmetic

- Reverse mathematics uses Z_2 to isolate the exact axioms on which well-known theorems from mathematics rely.
- For example, take the Hahn-Banach theorem (analytic form) which says
- Let E be a normed linear space over $\mathcal{R}$, $F \subseteq E$ a vector subspace, $\phi : F \to \mathcal{R}$ such that ϕ is linear and bounded. Then $\exists\ \tilde{\phi} : E \to \mathcal{R}$ such that $\tilde{\phi}$ is linear, $\tilde{\phi}|_F = \phi$ and $||\tilde{\phi}|| = ||\phi||$
- If we scrutinize the proof we observe that only sub-additivity, positive homogeneity of ϕ has been used. Hence we can restate the theorem in more general setting -
- Let $F \subseteq E$ be a normed linear space over $\mathcal{R}$, p is a subadditive, positively homogeneous function on E, $\phi \in L(F,\mathcal{R})$, $\phi(x) \leq p(x) \forall x \in F$. Then $\exists\ \tilde{\phi} \in L(E,\mathcal{R})$ such that $\tilde{\phi}(x) \leq p(x)$ and, $\tilde{\phi}|_F = \phi$
- These types of analysis are usually done in the Z_2 environment and with great success

Second Order Set Theory

- Till now we treated set theory as a first order logic construct. But in originality when Zermelo introduced the axioms, he did it in the second order logic fashion
- In paricular, his Separation Axiom is

$$\forall x \forall X \exists y \forall z (z \in Y \leftrightarrow (z \in x \land X(z)))$$

and the Replacement Axiom is

$$\forall x \forall F \ \exists y \forall z (z \in y \leftrightarrow \exists u (u \in x \land z = F(u)))$$

- Second-order ZFC, denoted by ZFC^2, is simply the received from first order ZFC with the Separation Schema replaced by the above single Separation Axiom, and the Replacement Schema replaced by the above single Replacement Axiom

Second Order Set Theory

- Second order set theory decides the Continuum Hypothesis, in the sense that, either it is a validity or not, although it can't decide which one. But it completely sides with one of the cases.
- Many set theorists think that the concept of set is definitive enough to decide eventually also C.H. even if ZFC does not decide it.
- Likewise, we may argue that the concept of second-order semantics is definitive enough to decide C.H. even if the current axioms of second-order logic cannot do it

THE END

THANK YOU