An Alternate Proof of Compactness Theorem & Corollaries

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Compactness Theorem

Γ is satisfiable if and only if all Γ^{fin} are satisfiable

*Some Landmark equivalent results:

- Weak Axiom of Choice
- Boolean Prime Ideal Theorem
- Banach -Alaoglu
- Ultrafilter lemma

Godel's Completeness Theorem

If *T* is any FOL-theory, and φ is any valid sentence in *T*, then T can be proven from the axioms
"Anything true in all models is provable"

$\Gamma \vdash \varphi$ if $\Gamma \vDash \varphi$

- The Generalised completeness theorem ($\Gamma \vDash \varphi$ iff $\Gamma \vdash \varphi$)
- Corollary: Compactness theorem
- *Compactness and Completeness are equivalent

Thm: If Γ is finitely satisfiable then Γ is satisfiable

Contrapositive: If Γ is not satisfiable then Γ is not finitely satisfiable (there exists a finite subset of Γ such that it is not satisfiable)

Assume that Γ is not satisfiable

$\Gamma \models \neg\varphi \land \varphi$ (Vacuously)

From Godel's Completeness Theorem, we can infer that $\neg\varphi \land \varphi$ can be deduced from finite elements from Γ. Call this subset of Γ, Ω.

From Soundness Theorem, we have $\Omega \models \neg\varphi \land \varphi$

Ω is not satisfiable!

■

Contrast

- Henkin's construction bypassed the concept of deductive consequences.

- Henkin's construction yielded a stronger result:

Compactness + Upward Lowenheim Skolem+Downward Lowenheim Skolem Theorems

Applications of Compactness Theorem

Thm: Any infinite graph is four colourable

Some prerequisites:

- *K-colourability* on a finite graph is a *firstorderisable* property.
- **Four Colour Theorem:** Any finite planar graph is four colourable

Proof: Consider any infinite planar graph *G=(V,E)*. Let **T** be an FOL theory(set of FOL formulas) defined to be the set of formulas characterising the four colourability of every finite subgraph of *G*.

T is finitely satisfiable. (Four colour Theorem).

Thm: Any infinite graph is four colourable

Hence **T** is satisfiable, i.e., there exists a model (a choice of assignment of colours to each vertex) such that the graph G is four colourable. $\blacksquare$

Note: This is the De Bruijn Erdos Theorem. Mycielski(1961) showed the equivalence between De Bruijn Erdos Theorem and Boolean Prime Ideal Theorem (which implies that De Bruijn Erdos Theorem and Compactness Theorem are equivalent)

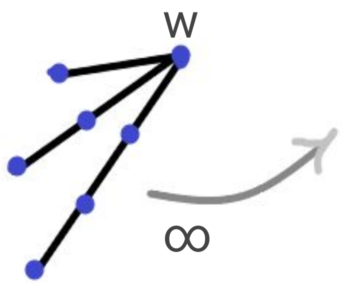
Konig's Lemma(1927)

Let G be a connected, locally finite and infinite graph. Then Konig's lemma asserts that G has a **ray** (a simple path that starts from one vertex and continues infinitely).

Seems obvious, right?

Consider the following non-example:

Here the vertex w emits infinite simple paths of lengths $n \in \mathbb{N}$.



It can be seen that there are no rays in this graph.

Konig’s Lemma(1927)

The subtlety lies in distinguishing between “there exist paths of all arbitrary lengths” and “there exists an interminable path”.

Proof: Instead of proving the theorem for any arbitrary graph, we can prove it for trees WLOG. Fix the root of the tree.

Given a tree T, define the predicate P_x to be “vertex x is chosen”.

We now build our Γ:

At least one vertex is chosen at each level

F_n: $\bigvee^{k}_{i=1} P_{n\_i}$ where n_i, $i \in [k]$ is the set of vertices at level n.

Konig's Lemma(1927)

At most one vertex is chosen

G_n: $\bigwedge^k_{i=1} \bigwedge^k_{j=i+1} (P_{n\_i} \wedge \neg P_{n\_j})$

A vertex is picked only if its ancestor has been picked

H_{xy}: $P_y \rightarrow P_x$ where x is the ancestor y.

Γ defined as $\{F_n \mid n \in \mathbb{N}\} \cup \{G_n \mid n \in \mathbb{N}\} \cup \{H_{xy} \mid x,y \in T$ and x is the ancestor of y$\}$

If Γ is satisfiable, then there is a ray in T.

Now consider an arbitrary $\Gamma^{fin} \subseteq \Gamma$. Since it contains finite occurrences of P_x, it contains finite occurrences of vertices of T.

Let v be the vertex of highest level (k) amongst them. Define

$\Gamma_k = \{F_n | n \in [k]\} \cup \{G_n | n \in [k]\}$

$\cup \{H_{xy} | x,y \in T$, x is the ancestor of y and $Lev(x), Lev(y) \leq k\}$

Observe:

1. Γ_k is finite.
2. $\Gamma^{fin} \subseteq \Gamma_k$
3. Γ_k is satisfiable implies Γ^{fin} is satisfiable.

Claim: Γ_k is satisfiable.

$\Gamma_k = \{F_n | n \in [k]\} \cup \{G_n | n \in [k]\}$

$\cup \{H_{xy} | x,y \in T$, x is the ancestor of y and Lev(x), Lev(y)$\leq$k} satisfiable

Iff there exists a path of length k starting from the root node.

Since T is an infinite tree and w has finite degree, we can always find such a path. Hence Γ_k is satisfiable.

Hence by our observation(3.), any arbitrary finite subset of Γ is satisfiable.

By Compactness theorem, Γ is satisfiable! ■

Every Field has an algebraic closure

Let F be a field. Consider $P_{f(x)}$: f(x) splits (f(x) is non constant). We have shown that Field Axioms are firstorderisable in Assignment 1.

Define Γ ={field axioms} (1)

$\cup$ {$P_{f(x)}$|f(x) is a non constant polynomial in F} (2)

$\cup$ {$\exists x(x=k) \mid k \in F$} (3)

Consider $\Gamma^{fin} \subseteq \Gamma$. Γ^{fin} contains finite elements of the second kind. Hence, a model satisfying Γ^{fin} would be $\mathcal{M}$=(D,I,G), D=F($\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n$)

where α_i are the roots of the polynomials occurring in sentences of type (2) in Γ^{fin} .

Every Field has an algebraic closure

Hence every finite subset of Γ is satisfiable. By compactness, Γ is satisfiable.

Hence there exists a model on the domain L(where L has been endowed field structure) such that $\mathcal{M}_L$ satisfies Γ.

Consider all the elements of L that are algebraic over F. Call this set F^*.

By borrowing a direct result from commutative algebra, we see that F^* is a field too.

Hence F^* is an algebraic closure of F

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Bibliography

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Thank You