Logic for Computer Science	08/01/2024-10/01/2024

Lecture 1 and 2: Introduction to First-order logic

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1 Introduction to logic - Lecture 1

Mathematical logic is widely known as the *calculus* of computer science. This is because logic plays a fundamental role in areas of computer science such as computability theory, algorithms (complexity), programming languages (semantics), databases (relational algebra), automated reasoning and the likes.

Definition of logic : Logic is the study of formal systems of reasoning and of methods of attaching meaning to them.

1.1 A brief history of logic

Logic was originally studied by the ancient Greek sophists so as to devise a set of rules to determine who wins a particular debate. The subject of logic essentially deals with the ability to distinguish what is true from what is false.

1.1.1 Early years of logic

Originally, logic dealt with deducing the correctness of arguments in natural language made by humans.For instance, let's take the following example :

- All men are mortal.
- Aristotle is a man.
- Therefore, Aristotle is mortal.

However, natural language representation turns out to be very ambiguous. Let's take the word "any".In the sentence

Eric doesn't believe that Mary can pass *any* test.

it could be taken to mean either "all" or "one". Another problem was that it led to many paradoxes. For instance, if we consider this example, which is famously known as the *Liar's paradox*

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This sentence is a lie.

If it were true, then it must be a lie, as it says, but this is impossible. Likewise, if it were a lie, then what it says is true!

Other examples are:

The Sophist's Paradox : A Sophist is sued for his tuition by his school. He argues that he must win, since, if he loses, the school didn't educate him well enough, and doesn't deserve the money. The school argues that he must lose, since, if he wins, he was educated well enough and therefore should pay for it.[1]

These paradoxes, along with the ambiguities of natural language, eventually led to the effort to formulate logic in a **symbolic language**.

1.2 Some examples of languages in different theories

- Let's consider the following statement in group theory: For every x there exists y such that $x.y = e$. where $(.)$ stands for the binary group operation and e is the identity element. Then the statement can be represented by $\forall x \exists y (x.y = e)$.

- Let's consider two statements from set theory :

  1. $\forall x \forall y \exists z (x \in z \wedge y \in z)$.

  2. $\neg \exists x \forall y (y \in x)$.

  The first statement is a symbolic representation of the statement *Given any two sets x and y, there is a set z that contains both x and y"; the second statement means that "There is no set x that contains all sets y.*

Thus, we can observe that the language for a theory should have *variables* to represent the objects of study, for instance, sets, groups etc and some *logical symbols* like $\exists$ (there exists/existential quantifier), $\forall$ (for all/universal quantifier), $\neg$ (negation), $\wedge$ (logical AND) and $=$ (equality) and some *non-logical symbols* representing specific concepts of a particular theory (for instance, the binary group operation $(.)$ and the identity element e in the theory of groups). Now, let's formally define the semantics of a first-order language.

1.3 First Order language

A **first-order language** L consists of two types of symbols: **logical symbols** and **non-logical symbols**. Logical symbols consist of a sequence of variables $x_0, x_1, x_2, ...$; **logical connectives** : $\neg$ (negation) and $\vee$ (disjunction), a **logical quantifier** $\exists$ (existential quantifier) and the **equality symbol** $=$.Depending on the theory under study, the set of non-logical symbols (represented by the tuple $(\mathcal{C}, \mathcal{F}, \mathcal{P})$) can be categorized into :

- (An empty or nonempty) Countable collection of constant symbols $\mathcal{C}$.
- (An empty or nonempty) Countable collection of n-ary function symbols $\mathcal{F}$, where for each $f \in \mathcal{F}$, $\#(f)$ gives us the arity of the function symbol f.
- (An empty or nonempty) Countable collection of n-ary function symbols $\mathcal{P}$, where for each $p \in \mathcal{P}$, $\#(p)$ gives us the arity of the function symbol p.

Some examples of first-order language are as follows -

1. The language for set theory has only one non-logical symbol: a binary relation symbol $\in$ for *belongs to*.
2. The language for group theory has a constant symbol e (for the identity element) and a binary function symbol $\cdot$ (for the group operation).
3. The language for the theory of rings with identity has two constant symbols 0 and 1 and two binary function symbols $+$ and $\cdot$.

2 Lecture 2

Let us first recall the definitions of *Terms* and *Formulas*.

Definition 2.1 (Term) *Terms are defined as:*

- *Any variable is a term*
- *Any constant symbol is a term*
- *Any f^n is an n-ary function symbol, and $t_1, t_2, ..., t_n$ are terms, then $f^n(t_1, t_2, ..., t_n)$ is a term*

Definition 2.2 (Primitive/Atomic Formula) *Primitive/Atomic Formulas are defined as:*

- *If t_1 and t_2 are terms, then $t_1 = t_2$ is a primitive formula.*
- *If P^n is an n-ary predicate symbol, and $t_1, t_2, ..., t_n$ are terms, then $P^n(t_1, t_2, ..., t_n)$ is a primitive formula.*

Definition 2.3 (Formula) *Primitive/Atomic Formulas are defined as:*

- *If Φ and Ψ are formulas and $\cdot$ is a binary logical connective, then $\Phi \cdot \Psi$ is a formula.*
- *If Φ is a formula and x is a variable, then $\exists x \Phi$ and $\forall x \Phi$ are formula.*

At first glance, we can think of terms and formulae as well-formed expressions that evaluates to a value in domain and *True* or *False*,respectively, given an interpretation or semantics of the logic we are dealing with. However, the notion of formula being evaluated to True *or* False is intrinsically tied with the Principle of Bivalence.

Definition 2.4 (Principle of Bivalence) *Given a proposition P, it has exactly one truth value, either True or False.*

This idea is closely associated with the notions of Law of Noncontradiction and Law of the Excluded Middle.

Definition 2.5 (Law of Noncontradiction) *Given any proposition P, $\neg(P \wedge \neg P)$ holds.*

Definition 2.6 (Law of Excluded Middle) *Given any proposition P, $P \vee \neg P$ holds.*

In plain terms, law of noncontradiction ensures that a statement and it's contradiction can not be true simultaneously. and the law of excluded middle ensures that either a statement or it's contradiction must be true. Thw two statements $\neg(P \ \& \ \neg P)$ and $P \vee \neg P$ might seem to be equivalent statements in terms of De Morgan's Laws. However, observe that De Morgan's law comes in the paradigm of Propositional Logic which itself satisfies these two laws. De Morgan's law does not hold, for example, in Institutional Logic, where the Law of Excluded Middle does not hold.

Looking beyond the formulations, let us take an example where we can appreciate the principle of bivalence and associated law. A classic example comes from Aristotle where he raised the question regarding the validity of principal of bivalence regarding statements based on future contingents. Consider the statement:

P : There will be a naval battle tomorrow.

As of today, Aristotle claims that the statement is neither true nor false, denying the validity of the law of excluded middle for statements based on future contingents. One of the approaches to resolve this has been through the paradigm of many valued logic. Consider for example the *Łukasiewicz Logic* proposed by polish logician Jan Łukasiewicz where a proposition can take three truth values: *True*, *False* and *As Yet Undetermined*; leaving space open for future contingents through the third option. Note that in this system, although the law of noncontradiction holds, the law of excluded middle does not hold, nor does the principle of bivalence. The issues of future contingents, have also been dealt in terms of temporal logic. For an example related to the validity of the law of noncontradiction, we can refer to *Logic of Paradox* proposed by the Argentinian logician Florencio González Asenjo.

For sake of completeness, we also give the statement of Law of Identity which, along with law of noncontradiction and law of excluded middle, forms the *three laws of thought*.

Definition 2.7 (Law of Identity) *Given any proposition P, $P = P$ holds.*

In first-order logic with identity (*The case we have considered*), this identity (or equality) is treated as a logical constant and its axioms are part of the logic itself. Under this setup, the law of identity becomes logical truth. In first-order logic without identity, identity is treated as an interpretable predicate and its axioms are supplied by the theory. This allows a broader equivalence relation to be used that may allow $a = b$ to be satisfied by distinct individuals a and b. Under this convention, a model is said to be normal when no distinct individuals a and b satisfy $a = b$. Before ending the discussion, we state the three laws of thought as stated by Bertrand Russell [2]:

- **Law of Identity:** 'Whatever is, is.'
- **Law of Noncontradiction:** 'Nothing can both be and not be.'
- **Law of Excluded Middle:** 'Everything must either be or not be.'

3 References

References

[1] H.B. Enderton. *A Mathematical Introduction to Logic*. Elsevier Science, 2001.

[2] Bertrand Russell. *The Problems of Philosophy*. Barnes & Noble, New York, 1912.