Lecture 13

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March 27, 2024

In this lecture, we will explore a corollary of the proposition on FOL equivalence of Isomorphic structures (with derived assignments) and do some preliminary work in establishing the proof of the Soundness Theorem.

Propostion 1. Let L be a first order language and A be an L structure. Let R be an n-ary relation on D_A such that R is definable on L. Let h be an automorphism on A. Then: $(a_1, a_2, \ldots a_n) \in R$ iff $(h(a_1), h(a_2), \ldots, h(a_n)) \in R$

Proof. Let $\phi(x_1, x_2, \ldots, x_n)$ be a formula (where $x_1, x_2, \ldots, x_n$ are the free variables) that defines the relation R. Then:
$(a_1, a_2, \ldots a_n) \in R$
iff
$A_{[x_1 \to a_1, x_2 \to a_2, \ldots x_n \to a_n]} \vDash \phi$
iff(two isomorphic models whose assignments are connected by the isomorphism in question are FO equivalent)
$A_{[x_1 \to h(a_1), x_2 \to h(a_2), \ldots x_n \to h(a_n)]} \vDash \phi$
iff
$(h(a_1), h(a_2), \ldots h(a_n)) \in R$

□

Example 1. Consider example 2 under definability of relations in Lecture 12. We showed that $\{b\}$ as a subset of G was not definable. This fact can also be seen via a straightforward application of the proposition above. Consider a structure automorphism $h : (G, I_A) \to (G, I_B$ where $h(a) = a, h(b) = c, h(c) = b$. Furthermore define I_B on constants, functions and predicates as below:
1. $I_B(c) = h(I_A(c))$ for all $c \in \mathcal{C}$
2.$I_B(f)(a_1, a_2, ..., a_n) = I_A(f)(h(a_1), h(a_2), ..., h(a_n))$ for all $f \in \mathcal{F}$
3.$I_B(p) = \{(h(a_1), h(a_2), ..., h(a_n) | (a_1, a_2, ..., a_n) \in I_A(p)\}$ for all $p \in \mathcal{P}$

h is clearly a set bijection, and by the construction of I_B it is also a structure homomorphism. Hence h is a bonafide structure automorphism.
From Proposition 1, If $R = \{b\}$ is a definable relation then $b \in R \implies h(b) \in R \implies c \in R$ which is a contradiction. Hence $R = \{b\}$ is not a definable relation.

Example 2. . Consider the structure $(\mathbb{R}, <)$. We will show that $\mathbb{N} \subset \mathbb{R}$ is not definable in $(\mathbb{R}, <)$. Consider the automorphism defined by $h(x) = x^3$ where interpretation on the co-domain is canonically defined (see previous example). By way of Proposition 1, we have $n \in \mathbb{N}$ iff $h(n) \in \mathbb{N}$ assuming $\mathbb{N}$ is definable. But there exist $n \notin \mathbb{N}$ but $n^3 \in \mathbb{N}$

Compactness theorem asserts that if $\Gamma \vDash \phi$ then there exists $\Gamma_0 \subseteq_{fin} \Gamma$ such that $\Gamma_0 \vDash \phi$. We are naturally motivated to ask the following questions about this result:

What is Γ_0? Can we concoct Γ_0 for any Γ?
How do we treat mathematical reasoning used in proving results in the conventional sense of the word 'prove'?

Deductive Consequence Relation
Deductive consequence relations can be defined in various ways, e.g., Hilbert-Style axiomatization, Gentzen's sequent calculus and Gentzen's natural deduction. In this course we will adopt Hilbert-Style axiomatization as our mode of definition.

Definition 1. Let Γ be a set of formulas and ϕ be a formula. ϕ is said to be a deductive consequence of Γ, i.e., $\Gamma \vdash \phi$ if there is a finite sequence of formulas $\phi_1, \phi_2, ..., \phi_n$ such that:
1.ϕ_n is ϕ
2. each ϕ_i

- is either a member of Γ
- is an *axiom*
- obtained by the application of some *rule of inference.*

Definition 2. Axioms are a certain subset of formulas in the language

Definition 3. A rule of inference is a subset of $\mathcal{P}(\mathcal{L}) \times \mathcal{F}(\mathcal{L})$ where $\mathcal{L}$ denotes the language under consideration. We write them as $\frac{\psi_1, \psi_2, ..., \psi_k (premise)}{\psi (conclusion)}$

Theorem 1. If $\Gamma \vdash \psi$ then $\Gamma \vDash \psi$ **(Soundness Theorem)**

The converse of this theorem also holds:

Theorem 2. If $\Gamma \vDash \psi$ then $\Gamma \vdash \psi$ **(Completeness Theorem)**

In this lecture, we will endeavour to outline a proof for the Soundness Theorem. A natural consideration emerges in this regard:

Proving which properties of axioms and rules of inference will aid us in proving the Soundness Theorem?

From the definition of deductive consequence it follows that if $\Gamma \vdash \phi$ then $\Gamma \vdash \phi_i$ where $i \in [n]$. We induct on the length (n) of this finite sequence.

Definition 4. The sequence $\phi_1, \phi_2, ..., \phi_n = \phi$ as described in the definition of a rule of inference is called the proof of ϕ from Γ.

Our proof boils down to inducting on the length of the proof of ϕ from Γ
Base Case: For the base case, where $n = 1$, we have $\phi_1 = \phi$ If

- $\phi_1 \in \Gamma$, then $\Gamma \vdash \phi$
- ϕ_1 is an axiom. To show that $\Gamma \vdash \phi$, it suffices to show that ϕ_1 is a validity.**(To show: Axioms are validities)**

Induction Hypothesis: Assume the proof holds for all $n \leq m, m \in \mathbb{N}$
For the induction step, where $n = m + 1$, consider ϕ_n. Cases when ϕ_n is in Γ or is an axiom have already been treated in the base case. So consider ϕ_n obtained by some rule of inference, say $\frac{\psi_1,\psi_2,..,\psi_k}{\psi=\phi}$ where $\psi_1, \psi_2, .., \psi_k$ are one of the following

- axioms
- member of Γ
- derived from the rules of inference.

For the first two cases $\psi_i \models \Gamma$ from the base case.
When ψ_i is itself derived from a rule of inference, we see that the length of its proof in Γ is less than n. Hence, by I.H. $\psi_i \models \Gamma$. We have, thus, shown for all $i \in [k]$ that $\psi_i \models \Gamma$ Suppose we can prove that **any model that satisfies the premises of a rule, also satisfies the consequence of the rule.** It immediately follows that $\psi \models \Gamma$.
Overall, we see that soundness theorem follows if we prove

- Axioms are validities
- Any model that satisfies the premises of a rule, also satisfies the consequence of the rule(Rules preserve consequences).

We will prove these assertions in the next lecture.