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Lecture 14: Classical Propositional Logic

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1 Introduction

We develop a simpler but important form of logic called *propositional logic*. The main objective of propositional logic is to formalize reasoning involving logical connectives $\vee$ and $\neg$ only.

A language for propositional logic has a nonempty set of variables and logical symbols $\vee$ and $\neg$ alone. Here variables stand for propositions rather than elements of a set. Then we consider the smallest set of expressions $\mathcal{F}$, to be called formulas, of the language that contains all variables and that is closed under $\neg$ and $\vee$. For instance, our set of variables could be the set of all elementary formulas of a first-order language L. Then $\mathcal{F}$ will coincide with the set of all formulas of L. We may think of variables of L to be some simple statements and formulas that can be made using variables and logical connectives. In this setting it is more important to examine the truth of a formula in terms of its subformulas.

2 Syntax of Propositional Logic

Thus *language* for a propositional logic L consists of

(i) **variables:** a nonempty set of symbols, and
(ii) **logical connectives:** $\neg$ and $\vee$.

L is denoted as the language of a propositional logic. A finite sequence of symbols of L is called an expression in L.

Let $\mathcal{F}$ be the smallest set of expressions in L that contains all variables and that contains the expression $\neg A$ whenever $A \in \mathcal{F}$, and contains $\vee AB$ whenever A and B are in $\mathcal{F}$. The expressions belonging to $\mathcal{F}$ are called **formulas** of L.

Example 2.1 *Let A stand for the statement "Humidity is high," B for "It will rain this afternoon," and C for "It will rain this evening." Let A, B, C be all the variables of L. Then the formula $A \to (B \vee C)$ stands for the statement "If humidity is high, it will rain this afternoon or this evening."*

3 Semantics of Propositional Logic

A **truth valuation** or an **interpretation** or a **structure** of L is a map v from the set of all variables of L to $\{T, F\}$.

Let v be an interpretation of L. We extend v (and denote the extension by v itself) to the set of all formulas by induction as follows:

$$v(\neg A) = T \text{ if and only if } v(A) = F$$

and

$$v(A \vee B) = T \text{ if and only if } v(A) = T \text{ or } v(B) = T.$$

If $v(A) = T$, we say that A is true in the structure v or that v satisfies A. Otherwise, A is said to be false in the structure.

Note that the truth value $v(A)$ of a formula A depends only on the variables occurring in A.

Let $\mathcal{A}$ be a set of formulas of L. An interpretation v is called a **model** of $\mathcal{A}$ if every $A \in \mathcal{A}$ is true in v. In this case we write $v \vDash \mathcal{A}$. If $\mathcal{A}$ has a model, we say that $\mathcal{A}$ is **satisfiable**.

Let A, B be formulas and $\mathcal{A}$ a set of formulas.

(i) We say that A is a **tautological consequence** of $\mathcal{A}$, and write $\mathcal{A} \vDash A$ if A is true in every model v of $\mathcal{A}$.

(ii) If A is a tautological consequence of the empty set of formulas, we say that A is a **tautology** and write $\vDash A$. Thus, A is a tautology if and only if $v(A) = T$ for every truth valuation v of L.

(iii) If $A \leftrightarrow B$ is a tautology (i.e., if $v(A) = v(B)$ for all truth valuations v), we say that A and B are **tautologically equivalent** and write $A \equiv B$.

4 Some Rules and Proofs

We define a proof in propositional logic. To define a proof syntactically, we fix some tautologies and call them **logical axioms**. Further, we fix some **rules of inference**. There is only one class of logical axioms, called **propositional axioms**. These are formulas of the form $\neg A \vee A$.

Rules of inference of the propositional logic are

(a) **Expansion Rule.** Infer $B \vee A$ from A.
(b) **Contraction Rule.** Infer A from $A \vee A$.
(c) **Associative Rule.** Infer $(A \vee B) \vee C$ from $A \vee (B \vee C)$.
(d) **Cut Rule.** Infer $B \vee C$ from $A \vee B$ and $\neg A \vee C$.

The rule for eliminating implication There is one rule to introduce $\rightarrow$ and one to eliminate it. The latter is one of the best known rules of propositional logic and is often referred to by its Latin name *modus ponens*.

This rule states that, given ϕ and knowing that ϕ implies ψ, we may rightfully conclude ψ. In our calculus, we write this as

$$\frac{\phi \quad \phi \rightarrow \psi}{\psi}$$

Now, let's look at a hybrid rule which has the Latin name *modus tollens*. It is like the $\rightarrow e$ rule in that it eliminates an implication. Suppose that $p \rightarrow q$ and $\neg q$ are the case. Then, if p holds we can use $\rightarrow e$ to conclude that q holds. Thus, we then have that q and $\neg q$ hold, which is impossible. Therefore, we may infer that p must be false. But this can only mean that $\neg p$ is true. We summarise this reasoning into the rule modus tollens, or MT for short:

$$\frac{\phi \to \psi \quad \neg\psi}{\neg\phi} MT$$

Again, let us see an example of this rule in the natural language setting:

'If Abraham Lincoln was Ethiopian, then he was African. Abraham Lincoln was not African; therefore he was not Ethiopian.'

4.1 Backus Naur Form

Definition 4.1 *The well-formed formulas of propositional logic are those which we obtain by using the construction rules below, and only those, finitely many times:*

atom: Every propositional atom $p, q, r, \ldots$ and $p_1, p_2, p_3, \ldots$ is a well-formed formula.
$\neg$: *If ϕ is a well-formed formula, then so is $(\neg\phi)$.*
$\wedge$: *If ϕ and ψ are well-formed formulas, then so is $(\phi \wedge \psi)$.*
$\vee$: *If ϕ and ψ are well-formed formulas, then so is $(\phi \vee \psi)$.*
$\to$: *If ϕ and ψ are well-formed formulas, then so is $(\phi \to \psi)$.*

Note that the condition 'and only those' in the definition above rules out the possibility of any other means of establishing that formulas are well-formed. Inductive definitions, like the one of well-formed propositional logic formulas above, are so frequent that they are often given by a defining grammar in **Backus Naur form (BNF)**. In that form, the above definition reads more compactly as

$$\phi ::= p|(\neg\phi)|(\phi \wedge \phi)|(\phi \vee \phi)|(\phi \to \phi)$$

where p stands for any atomic proposition and each occurrence of ϕ to the right of $::=$ stands for any already constructed formula.

4.2 Conjunctive Normal Form

Definition 4.2 *A literal L is either an atom p or the negation of an atom $\neg p$. A formula C is in **conjunctive normal form (CNF)** if it is a conjunction of clauses, where each clause D is a disjunction of literals:*

$$L ::= p|\neg p$$

$$D ::= L|L \vee D$$

$$C ::= D|D \wedge C$$

Lemma 4.3 *A disjunction of literals $L_1 \vee L_2 \vee \ldots \vee L_m$ is valid iff there are $1 \leq i, j \leq m$ such that L_i is $\neg L_j$.*

Proof. If L_i equals $\neg L_j$, then $L_1 \vee L_2 \vee \ldots \vee L_m$ evaluates to T for all valuations. For example, the disjunct $p \vee q \vee r \vee \neg q$ can never be made false. To see that the converse holds as well, assume that no literal L_k has a matching negation in $L_1 \vee L_2 \vee \ldots \vee L_m$. Then, for each k with $1 \leq k \leq n$, we assign F to L_k, if L_k is an atom; or T, if L_k is the negation of an atom. □

For example, the disjunct $\neg q \vee p \vee r$ can be made false by assigning F to p and r and T to q.

Lemma 4.4 *Let ϕ be a formula of propositional logic. Then ϕ is satisfiable iff $\neg\phi$ is not valid.*

Proof. First, assume that ϕ is satisfiable. By definition, there exists a valuation of ϕ in which ϕ evaluates to T; but that means that $\neg\phi$ evaluates to F for that same valuation. Thus, $\neg\phi$ cannot be valid. Second, assume that $\neg\phi$ is not valid. Then there must be a valuation of $\neg\phi$ in which $\neg\phi$ evaluates to F. Thus, ϕ evaluates to T and is therefore satisfiable. □

5 Conclusion

Logic has a long history stretching back at least 2000 years, but the truth value semantics of propositional logic presented in this and every logic textbook today was invented only about 160 years ago, by G. Boole.

Natural deduction was invented by G. Gentzen, and further developed by D. Prawitz. Other proof systems existed before then, notably axiomatic systems which present a small number of axioms together with the rule modus ponens (which we call $\rightarrow e$). Proof systems often present as small a number of axioms as possible; and only for an adequate set of connectives such as $\rightarrow$ and $\neg$. This makes them hard to use in practice. Gentzen improved the situation by inventing the idea of working with assumptions (used by the rules $\rightarrow i$, $\neg i$ and $\vee e$) and by treating all the connectives separately.

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Lecture 15: Completeness Theorem	
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1 Prooof of Completeness Theorem

With the continuation of the completeness theorem in the previous class, we have to show that $\Gamma \vdash \phi \to \psi$. In the last class, we have seen that we will prove this by applying induction on the length of a derivation/proof of ψ from $\Gamma \cup \{\phi\}$. In the last class we have covered the base case. Now we continue with the rest of the part of this proof.

- **Induction Hypothesis:-** Suppose the result holds for all derivation of the length $\leq$ n.

- **Induction Step:-** Suppose the length of the derivation of ψ from $\Gamma \cup \{\phi\}$ is n+1. Thus, by the definition of $\vdash$, Γ is either an axioms or a member of $\Gamma \cup \{\phi\}$ or obtained by some rule. All these cases are dealt except for the final case, where Γ is obtained by some rule. Now, only rule that we have considered is M.P. So, let us assume that Γ has been obtained by M.P., then there exists Γ_i and Γ_j in the derivation o the Γ from $\Gamma \cup \{\phi\}$, such that Γ_j is of the form $\Gamma_i \to \Gamma_j$. Then we have $\Gamma \cup \{\phi\} \vdash \Gamma_i$ and $\Gamma \cup \{\phi\} \vdash \Gamma_j \to \Gamma$. Then, by the induction hypothesis, we have,

$$\Gamma \vdash \phi \to \Gamma_i$$

and

$$\Gamma \vdash \phi \to (\Gamma_i \to \Gamma)$$

Now to show, $\Gamma \vdash \phi \to \psi$, We have

$$\begin{aligned} \Gamma \vdash &\phi \to \psi_i \\ &\phi \to (\psi_i \to \psi) \end{aligned}$$

Let us now consider an axiom:-

$$(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$$

We have the following derivation of $\phi \to \phi$ from Γ

$$\begin{aligned} \Gamma \vdash &1.\phi \to \psi_i (\text{I.H}) \\ &2.\phi \to (\Gamma_i \to \Gamma)(\text{I.H}) \\ &3.(\phi \to (\Gamma_i \to \Gamma)) \to ((\phi \to \psi_i) \to (\phi \to \psi))(\text{Axiom}) \\ &4.(\phi \to \psi_i) \to (\phi \to \psi)(\text{M.P. } 2, 3) \\ &5.\phi \to \psi(\text{M.P. } 1, 4) \end{aligned}$$

This completes the proof of the only-if part of the property 2.

- **Property 3**:- $\{ \phi, \neg\phi\} \models \psi$

To show $\{\phi, \neg\phi\} \vdash \psi$

now,

$$\begin{aligned}\{\phi, \neg\phi\} \vdash & 1.\phi(\text{Premise}) \\ & 2.\neg\phi(\text{Premise}) \\ & 3.\phi \to (\neg\psi \to \phi)(\text{Axiom}) \\ & 4.\neg\phi \to \phi(\text{M.P } 1, 3) \\ & 5.\neg\phi \to (\neg\psi \to \neg\phi)(\text{Axiom}) \\ & 6.\neg\psi \to \neg\phi(\text{M.P. } 2, 5)\end{aligned}$$

[Let us consider an axiom: $(\psi \to \phi) \to ((\neg\psi \to \neg\phi) \to \psi)$]

$$\begin{aligned}& 7.(\neg\psi \to \phi) \to ((\neg\psi \to \neg\phi) \to \psi) \\ & 8.(\neg\psi \to \neg\phi) \to \psi(\text{M.P. } 4, 7) \\ & 9.\psi(\text{M.P. } 6, 8)\end{aligned}$$

This complete the proof of $\{\phi, \neg\phi\} \vdash \psi$ Now, let us note that we can derive $\phi \to \phi$ from the axioms:-

1. $\phi \to (\psi \to \phi)$
2. $(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$

Exercise
Show that $\phi \to \phi$ can be derivable from the axioms:-

1. $\phi \to (\psi \to \phi)$
2. $(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$

So we have the following collection of axioms and rules:

Axioms and Rule:-

- Axioms:-

  1 $\phi \to (\psi \to \phi)$

  2 $(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$

  3 $(\neg\phi \to \psi) \to ((\neg\phi \to \neg\psi) \to \phi)$

- Rule:-

  (a) $\frac{\phi \quad \phi\to\psi}{\psi}$

This is the axiom system for the Classical Propositional Logic.

Now we need to show that:-

$$\text{If } \Gamma \models \phi \text{ then } \Gamma \vdash \phi$$

We will prove this in a contrapositive way. We assume that $\Gamma \nvdash \phi$ and show that $\Gamma \not\models \phi$. Now showing $\Gamma \not\models \phi$ is the same as showing $\Gamma \cup \{\neg\phi\}$ is satisfiable.

What are the steps for this proof:-

1. We introduce a notion of consistency. (A deduction theoretic concept)
2. $\Gamma \nvdash \phi$ iff $\Gamma \cup \{\neg\phi\}$ is consistent.
3. Every consistent set of formulas is satisfiable.

Now let us try to prove the above-mentioned steps.

1. Notion of consistency:-
   - A set of formulas Γ is said to be inconsistent if there is a formula ϕ such that $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$
   - A set of formula Γ is said to be consistent if it is not consistent.
2. **$\Gamma \nvdash \phi$ iff $\Gamma \cup \{\neg\phi\}$ is consistent.**
   - **If- part:-** Let $\Gamma \cup \{\neg\phi\}$ be consistent. To show that $\Gamma \nvdash \phi$. Suppose not, then $\Gamma \vdash \phi$. Now, $\Gamma \cup \{\neg\phi\} \vdash \neg\phi$ and $\Gamma \cup \{\neg\phi\} \vdash \phi$. Hence, $\Gamma \cup \{\neg\phi\} \vdash \phi$ is contradiction. Thus it makes the contradiction. So, $\Gamma \nvdash \phi$.
   - **Only if-part:-** Let $\Gamma \nvdash \phi$. To show that $\Gamma \cup \{\neg\phi\}$ is consistent. Let us assume that $\Gamma \cup \{\neg\phi\}$ is not consistent. Then for every some formula ψ, $\Gamma \cup \{\neg\phi\} \vdash \psi$ and $\Gamma \cup \{\neg\phi\} \vdash \neg\psi$. Then we have,

$$\begin{aligned}
\Gamma \vdash & 1. \neg\phi \to \psi \qquad \text{(D.T.)} \\
& 2. \neg\phi \to \neg\psi \qquad \text{(D.T.)} \\
& 3. (\neg\phi \to \neg\psi) \to ((\neg\phi \to \neg\psi) \to \phi) \qquad \text{(Axiom 3)} \\
& 4. (\neg\phi \to \neg\psi) \to \phi \qquad \text{(M.P. 1, 3)} \\
& 5. \phi \qquad \text{(M.P. 2, 4)}
\end{aligned}$$

   Hence, $\Gamma \vdash \phi$ makes a contradiction. Hence, $\Gamma \cup \{\neg\phi\}$ is consistent. This completes the proof.
3. **Every consistent set of formulas is satisfiable:-**
   (a) Extend the consistent set Γ, say, to a consistent set and complete set Δ, say.
   (b) Define a valuation. V_Δ (That is a model in CPL).
   (c) Prove that for all formulas ϕ, $V_\Delta \models \phi (V_\Delta = 1)$ iff $\phi \in \Delta$.

(a) Extend the consistent set Γ, to a consistent set and complete set Δ:- Let Γ be a consistent set of formulas. Let us enumerate all formulas:- $\{\phi_1, \phi_2, \phi_3, \cdots\}$. Now define

$$\Delta_0 = \Gamma$$

$$\Delta_{k+1} \begin{cases} \Delta_k \cup \{\phi_k\}, & \text{if it is consistent} \\ \Delta_k, & \text{Otherwise} \end{cases}$$

Let $\Delta = \cup_{n \geq 0} \Delta_n$. Therefore, if Δ is consistent and complete, then our claim is enoght to show the following lemma:-

Lemma 1 *If Γ is a consistent set of formulas then either $\Gamma \cup \{\phi\}$ or $\Gamma \cup \{\neg\phi\}$ is consistent, for any formula ϕ.*

Proof. Suppose not. Let ϕ be a formula such that $\Gamma \cup \{\phi\}$ and $\Gamma \cup \{\neg\phi\}$ are inconsistent. Since, $\Gamma \cup \{\phi\}$ is inconsistent there is a formula ψ such that $\Gamma \cup \{\phi\} \vdash \psi$ and $\Gamma \cup \{\phi\} \vdash \neg\psi$. So, by D.T., we have $\Gamma \vdash \phi \to \psi$ and $\Gamma \vdash \phi \to \neg\psi$. by axiom 3, it can be shown that $\vdash (\phi \to \psi) \to ((\phi \to \neg\psi) \to \neg\psi)$ Now,

$$\begin{aligned} \Gamma \vdash & 1. \phi \to \psi \\ & 2. \phi \to \neg\psi \\ & 3. (\phi \to \psi) \to ((\phi \to \neg\psi) \to \neg\phi) \qquad \text{(Theorem)} \end{aligned}$$

[**Theorem:-** A formula ϕ said to be a theorem if $\Phi \vdash \phi$, that is $\vdash \phi$.]

$$\begin{aligned} & 4. (\phi \to \neg\psi) \to (\neg\phi) \qquad \text{(M.P. 1, 3)} \\ & 5. \neg\phi \qquad \text{(M.P. 2, 4)} \end{aligned}$$

So, we have $\Gamma \vdash \neg\phi$

Similarly using the fact that $\Gamma \cup \{\neg\phi\}$ is inconsistent, we can show that $\Gamma \vdash \phi$.

So, $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$, that is Γ is inconsistent, a contradiction. □

i. Show that $\Gamma \vdash \phi$

ii. Every consistent and complete set of formulas forms a model set.

(b) Let Δ be a consistent and complete set of formulas extending the consistent set Γ we started with:

Define $V_\Delta : \mathcal{P} \to \{0, 1\}$ by:

$$V_\Delta(p) = 1 \text{ iff } p \in \Delta$$

(c) **Truth lemma:-** For all formulas ϕ $V_\Delta(\phi) = 1$ iff $\phi \in \Delta$.

We prove this by applying induction on the size of the formulas.

- **Base Case:-** $\phi =$ p. To show that $V_\Delta(p) = 1$ iff p $\in \Delta$. This follows from the definition of V_Δ.

- **Induction Hypothesis:-** Suppose the result holds for all formulas of the size $\leq$ n.
- **Induction Step:-** Suppose ϕ is a formula of size n+1.
  - **Case 1:-** $\phi = \neg\psi$. $V_\Delta(\phi) = 1$ iff $V_\Delta(\neg\psi) = 1$ iff $V_\Delta(\psi) = 0$ iff $\psi \notin \Delta$ by induction hypothesis. Iff $\neg\psi \in \Delta(\Delta$ being a model cut) iff $\phi \in \Delta$.
  - **Case 2:-** $\phi = \psi \to \chi$. $V_\Delta(\phi) = 1$ iff $V_\Delta(\psi \to \chi) = 1$ iff $V_\Delta(\psi) = 0$ or $V_\Delta(\chi) = 1$ iff $\psi \notin \Delta$ or $\chi \in \Delta$ (By Induction Hypothesis) iff $(\psi \to \chi \in \Delta)$ (Δ being a model set) iff $\phi \in \Delta$.

This completes the proof. So, we finish our completeness proof for CPL.