Logic for Computer Science January 24 2024

Lecture 5: On Consequence and Satisfiability

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1 Models and Theories:

Definitions:

1. **Semantic consequence:** For a formula ϕ and for a set of formulas Γ, we say that ϕ is a semantic consequence of Γ if for all models μ with $\mu \models \gamma$ for all $\gamma \in \Gamma$ imply $\mu \models \phi$. This is denoted by $\Gamma \models \phi$.

2. **Satisfiabile:** A formula ϕ is said to be satisfiable if there is a model μ such that $\mu \models \phi$. For a set of formulas Γ, Γ is satisfiable if there is a model of Γ.

   Now for a given set of formulas Γ, $\Gamma \not\models \phi$ iff there is a model μ such that $\mu \models \Gamma$ and $\mu \not\models \phi$ iff there is a model μ such that $\mu \models \Gamma$ and $\mu \models \neg\phi$ iff there is a model μ such that $\mu \models \Gamma \cup \{\neg\phi\}$ iff $\Gamma \cup \{\neg\phi\}$ is satisfiable.

3. In a first-order language given any formula ϕ, the set of all models satisfying ϕ is given by $Mod(\phi)$.
   $$Mod(\phi) = \{\mu : \mu \models \phi\}.$$
   For a set of formulas Γ, the set of all models satisfying Γ is given by $Mod(\Gamma)$.
   $$Mod(\Gamma) = \{\mu : \mu \models \Gamma\}.$$
   Let Γ_1, Γ_2 be two sets of formulas such that $\Gamma_1 \subseteq \Gamma_2$, then, $Mod(\Gamma_2) \subseteq Mod(\Gamma_1)$.

4. **Theory:** Let κ be a class of models. Then the theory of κ, denoted by $Th(\kappa)$ and is defined by
   $$Th(\kappa) = \{\phi : \mu \models \phi \text{ for all models } \mu \text{ in } \kappa\}.$$
   Let κ_1, κ_2 be two classes of models such that $\kappa_1 \subseteq \kappa_2$, then, $Th(\kappa_2) \subseteq Th(\kappa_1)$.

5. **Consequence of a set of formulas:** Let Γ be a set of formulas. Then consequence of Γ, denoted by $Con(\Gamma)$ is defined by $Con(\Gamma) = Th(Mod(\Gamma))$.

6. **Definability ;** A class of first order structure K is said to be first order definable if there is a set of sentences Γ such that $Mod(\Gamma) = K$.

Proposition 1.1 ***Let*** Γ ***be a set of formulas and*** ϕ ***be a formula. Then*** $\phi \in Con(\Gamma)$ ***iff*** $\Gamma \models \phi$***.***

Proof. Suppose $\Gamma \models \phi$ to show $\phi \in Con(\Gamma)$.
All models in $Mod(\Gamma)$ satisfy ϕ. $\phi \in Th(Mod(\Gamma))$. So, $\phi \in Con(\Gamma)$

Conversely, let, $\phi \in Con(\Gamma)$. To show, $\Gamma \models \phi$.
$\phi \in Th(Mod(\Gamma))$ implies $\mu \models \phi$ for all $\mu \in Mod(\Gamma)$. Thus $\Gamma \models \phi$. □

Properties of $Con(\Gamma)$:

1. $\Gamma \subseteq Con(\Gamma)$
2. If $\Gamma_1 \subseteq \Gamma_2$ then $Con(\Gamma_1) \subseteq Con(\Gamma_2)$.
3. $Con(Con(\Gamma)) = Con(\Gamma)$.

Consider a first-order language L with countably many predicate symbols $p_1^1, p_2^1, \cdots$ all having arity 1. Then to make Γ, given set of formulas satisfiable we need to define a model satisfying it. This is not unique. For example If $\Gamma = \{p_1^1 x\}$

- Choose $D = \{a, b\}$, $I(p_1^1) = \{a\}$ and define $G : V \to D$ such that $G(y) = a$ for all y.
- Again Choose $D = \{a, b\}$, $I(p_1^1) = \{a\}$ and define $G : V \to D$ such that $G(y) = a$ if $y = x$ and $G(y) = b$ if $y \neq x$.
- Choose $D = \mathbb{N}$, $I(p_1^1) = \{n \in \mathbb{N} : n \text{ is even}\}$ define $G : V \to D$ such that $G(y) = 2$ if $y = x$ and $G(y) = 0$ if $y \neq x$.

In all cases $G(x) \in I(p_1^1)$ and $(D, I, G) \models p_1^1 x$.
For any number $k \geq 2$ we can find an unsatisfiable formula of size k. For example

- $\Gamma = \{p_1^1 x, \neg p_1^1 x\}$ is an unsatisfiable set of size 2.
- $\Gamma = \{p_1^1 x, p_2^1 x \neg(p_1^1 x \wedge p_2^1 x)\}$ is an unsatisfiable set of size 3.
- $\Gamma = \{p_1^1 x, p_2^1 x \neg(p_1^1 x \wedge p_2^1 x)\}$ is an unsatisfiable set of size 4 and so on.

If Γ be an infinite set of formulas such that all finite subsets of Γ is satisfiable then Γ is also satisfiable.

Theorem 1.2 (Compactness Theorem for First Order Languages) *Let Γ be an infinite set of formulas. Then Γ is satisfiable iff every finite subset of Γ is satisfiable.*

Theorem 1.3 *The following are equivalent*

1. *If Gamma is finitely satisfiable then Gamma is satisfiable.*
2. *If $\Gamma \models \phi$ then there is a finite subset of $Gamma_0$ of Gamma such that $Gamma_0 \models \phi$.*

Proof. (2 $\Rightarrow$ 1) Let Γ be fin-sat. To show that Γ is sat. Suppose not then $\Gamma \models \phi$ for all formulas ϕ. There is a formula ψ such that $\Gamma \models \psi$ and $\Gamma \models \neg\psi$. So we have

$$\Gamma_1 \subseteq_{fin} \Gamma \text{ such that } \Gamma_1 \models \psi$$

$$\Gamma_2 \subseteq_{fin} \Gamma \text{ such that } \Gamma_2 \models \neg\psi$$

$$\text{So } \Gamma_1 \cup \Gamma_2 \models \psi \wedge \neg\psi \text{ or } \Gamma_1 \cup \Gamma_2 \subseteq_{fin} \Gamma$$

$\Gamma_1 \cup \Gamma_2$ is not satisfiable, a contradiction.

$(1 \Rightarrow 2)$ Let $\Gamma \models \phi$. To show that there exists a set of formulas Γ_0 such that $\Gamma_0 \subseteq_{fin} \Gamma$ and $\Gamma_0 \models \phi$. Suppose not. So for all $\Gamma_0 \subseteq_{fin} \Gamma$, $\Gamma_0 \models \neg\phi$. Or, for all $\Gamma_0 \subseteq_{fin} \Gamma$, $\Gamma_0 \cup \{\neg\phi\}$ is satisfiable. Then by (1), $\Gamma \cup \{\neg\phi\}$ is satisfiable. Then $\Gamma \not\models \phi$ i.e. a contradiction arises. Hence the result. □

Theorem 1.4 *Let* Γ *be a set of sentences having arbitrarily large set of finite models. Then* Γ *has an infinite model.*

Proof. Let $D = \{d_1, d_2, d_3, \dots\}$ be a countable collection of new constant symbols not occurring in Γ. Consider $\Delta = \Gamma \cup \{\neg(d_i = d_j) : i, j \in \mathbb{N}, i \neq j\}$. Now Γ is satisfiable and hence finitely satisfiable. Take any finite subset of $\{\neg(d_i = d_j) : i, j \in \mathbb{N}, i \neq j\}$. Such a finite set will be satisfiable in a model of Γ having that many distinct elements. So Δ is fin-sat. Hence by compactness theorem, Δ is sat. $\Gamma \subseteq \Delta$. So a model of Δ is also a model of Γ. i.e Γ also has an infinite model. □

Exercise : Let FIN be the class of finite structures. Then show that FIN is not first-order definable.

Reference :

1. A Course on Mathematical Logic by Sashi Mohan Srivastava