Lecture 6

Proof of Completeness Theorem

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***Theorem* 1.** *Let* Γ *be an infinite set of formulas. Them* Γ *is satisfiable if it is finitely satisfiable.*

It can be immediately concluded from the definition of satisfiability if Γ is satisfiable then it should also be finitely satisfiable. i.e, if $\exists$ a model $M \ni M \vDash \Gamma$, then $M \vDash \phi$ where ϕ is any finite subset of Γ.

Establishing the converse of the theorem, is however, not very straightforward. We will preempt a rather lengthy proof in steps, by proving some lemmas and introducing new definitions.

***Definition* 1.** Consider M to be a non-empty model. Define Δ_M to be the set of all formulas that the model M satisfies, i.e.,

$$\Delta_M := \{\phi : M \vDash \phi\}$$

We are very naturally prompted to ask what properties does Δ_M have given a model M. Using the theory from the previous lectures, the following key points can be inferred

1. Given any formula ϕ,

(a)Δ_M contains either ϕ or $\neg\phi$

(b)Δ_M cannot contain both

2.For any formulas ϕ and ψ we have the following consequences:

(a) $\phi \in \Gamma$ or $\psi \in \Gamma$ iff $\phi \vee \psi \in \Gamma$

(b) $\phi \in \Gamma$ and $\psi \in \Gamma$ iff $\phi \wedge \psi \in \Gamma$

(c) $\phi \rightarrow \psi \in \Gamma$ iff $\psi \in \Gamma$ whenever $\phi \in \Gamma$

(d)$\phi \longleftrightarrow \psi \in \Gamma$ iff $\phi \in \Gamma \Leftrightarrow \psi \in \Gamma$

***Definition* 2.** Consider a Γ to be any set of formulas. If Γ satisfies properties 1.(a),(b) and 2.(a)-(d) then Γ is called a **model set**

***Definition* 3.** A set of formulas Γ is said to be **complete** if for any formula ϕ either $\phi \in \Gamma$ or $\neg\phi \in \Gamma$.

***Lemma* 1.** The set of all formulas in first order language is countable.

Proof. In first order logic, formulas are expressions that result from a finitely many applications of the following rules;
1.Given any terms t_1 and t_2, $t_1 = t_2$ is a formula
2.Given any n-ary predicate symbol P, $P(t_1, t_2, \ldots t_n)$ is a formula
3.If ϕ is a formula then $\neg\phi$ is a formula
4.If ϕ and ψ are formulas then $\phi \rightarrow \psi$ is a formula. Similar rules apply to other binary logical connectives
5.If ϕ is a formula and x is a variable then $\forall x\phi$ is a formula. Similar rules apply for other quantifiers. In Lecture 2, we have stipulated that in first order order language the set of primitive predicates is a countable set. Given $n \in \mathbb{N}$, the formulas of size n is at least $\mathbb{N} \times \mathbb{N} \times \cdots$ (n times) $\times \mathbb{N}$ Hence the set of formulas of size n is a countable set. Hence the set of all formulas is countable (since countable union of countable sets is countable).

We can now begin delineating the three main ideas that subsume the proof of the converse.

***Claim* 1.** Every finitely satisfiable set of formulas can be extended to a finitely satisfiable and a complete set of formulas

***Claim* 2.** Every finitely satisfiable and a complete set of formulas is a model set

***Claim* 3.** Every model set is satisfiable

Claims 1 and 2 show that every finitely satisfiable set of formula Γ can be seen as a subset of larger finitely satisfiable and complete set of formulas, say Γ', which will be a model set

Claim 3 further asserts that this Γ' will be a satisfiable set of formulas. Hence there exists a model M that satisfies Γ' and by extension any of its subsets. In particular Γ is satisfiable, which completes the proof.

Proof. **(Claim 1)**

As the set of all formulas in any first order language in countable (Lemma1), we can enumerate them. Consider one such choice of enumeration:

$$\phi_0, \phi_1, \phi_2, \ldots$$

Let Γ be a finitely satisfiable set of forumulas. Consider the following construction of a sequence of formulas derived from Γ; starting with $\Gamma_0 = \Gamma$, and

$$\Gamma_{k+1} = \begin{cases} \Gamma_k \cup \{\phi_k\} & \text{if } \Gamma \cup \{\phi_k\} \text{ is finitely satisfiable} \\ \Gamma_k & \text{otherwise} \end{cases}$$

Hence, $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \ldots$. Define $\Delta := \bigcup_{k \in \mathbb{N}} \Gamma_k$. We will now show that Δ is a complete and finitely satisfiable set.

Claim(a): Δ is complete

It is enough to show that for any finitely satisfiable set of formulas Γ' either $\Gamma' \cup \{\phi\}$ is finitely satisfiable or $\Gamma' \cup \{\neg\phi\}$ is finitely satisfiable given any formula ϕ

We can begin by assuming Γ' is finitely satisfiable. Assume that $\Gamma' \cup \{\phi_k\}$ and $\Gamma' \cup \{\neg\phi\}$ are not finitely satisfiable for some formula ϕ_k.

Hence there exist non-empty(Lec4, HW3), satisfiable subsets $\Gamma'_1, \Gamma'_2 \subseteq_{fin} \Gamma'$ such that

$$\Gamma_1 = \Gamma'_1 \cup \{\phi\}$$
$$\Gamma_2 = \Gamma'_1 \cup \{\neg\phi\}$$

are not satisfiable Hence $\Gamma'_1 \vDash \neg\phi$ and $\Gamma'_2 \vDash \phi$. Therefore, $\Gamma'_1 \cup \Gamma'_2 \vDash \phi \wedge \neg\phi$. This implies that $\Gamma'_1 \cup \Gamma'_2$ is not satisfiable but this leads to a contradiction as $\Gamma'_1 \cup \Gamma'_2$ is a finite subset of Γ'.

Hence, claim proved.

Claim(b): Δ is finitely satisfiable

Suppose Δ is not finitely satisfiable. Then there is $\Delta' \subseteq_{fin} \Delta$ such that Δ' is not sat-

isfiable. But $\Delta' \subseteq \Gamma_k$ for some $k \in \mathbb{N}$ which contradicts the very construction of Γ_k sequence. Hence, claim proved.
This concludes the proof of Claim 1 □

We now proceed to prove Claim 2. We will show that any finitely satisfiable and complete set of formulas satisfies the properties 1. and 2.

Proof. Consider a finitely satisfiable and complete set of formulas Δ. 1.(a) follows immediately from the definition of completeness. Δ cannot contain ϕ and $\neg\phi$ because finite satisfiability would necessitate $\{\phi \cup \neg\phi\}$ to be satisfiable. Hence 1.(b) holds. Now we will show the validity of properties 2.(a)-(d).
Consider any two formulas ϕ and ψ

<u>$\phi \in \Gamma$ **or** $\psi \in \Gamma$ **iff** $\phi \vee \psi \in \Gamma$</u>

Consider $\phi \vee \psi \in \Delta$. If neither ϕ nor ψ are in Δ, then by completeness, $\neg\phi \in \Delta$ and $\neg\psi \in \Delta$, which implies $\{\phi \vee \psi, \phi, \neg\phi\}$ is satisfiable by finite satisfiability of Δ But the aforementioned set of formulas is unsatisfiable. Hence, either ϕ or ψ is in Δ.
Now for the converse, assume WLOG $\phi \in \Delta$. If $\phi \vee \psi \notin \Gamma$, then by the completeness of Δ we have $\neg(\phi \vee \psi) \in \Gamma$ But $\{\neg(\phi \vee \psi), \phi\}$ is not satisfiable, contradicts the finite satisfiability of Δ. Hence $\phi \vee \psi \in \Delta$.

<u>$\phi \in \Gamma$ **and** $\psi \in \Gamma$ **iff** $\phi \wedge \psi \in \Gamma$</u>

Consider $\phi \wedge \psi \in \Delta$. WLOG assume ϕ is not in Δ. Then by completeness, $\neg\phi \in \Delta$, which implies $\exists M$ such that $\{\phi \wedge \psi, \neg\phi\}$ is satisfiable by finite satisfiability of Δ But the aforementioned set of formulas is unsatisfiable. Hence, ϕ is in Δ.
Now for the converse, assume $\phi \in \Delta$ and $\psi \in \Delta$. If $\phi \wedge \psi \notin \Gamma$, then by the completeness of Δ we have $\neg(\phi \wedge \psi) \in \Gamma$ But $\{\neg(\phi \wedge \psi), \phi, \psi\}$ is not satisfiable, contradicts the finite satisfiability of Δ Hence $\phi \wedge \psi \in \Delta$.

<u>$\phi \rightarrow \psi \in \Gamma$ **iff** $\psi \in \Gamma$ **whenever** $\phi \in \Gamma$</u>

Assume $\phi \rightarrow \psi \in \Gamma$ holds. If $\phi \in \Gamma$ and $\psi \notin \Gamma$, then $\{\phi \rightarrow \psi, \phi, \neg\psi\}$ is satisfiable. This is a contradiction.
Now to show the converse, assume $\psi \in \Gamma$ whenever $\phi \in \Gamma$.

Case 1: $\phi \in \Gamma$

We have $\psi \in \Gamma$. If $\phi \rightarrow \psi \notin \Gamma$ Then $\{\neg(\phi \rightarrow \psi), \phi, \psi\}$ is satisfiable which can happen iff(Lecture-3) $\{\phi \wedge \neg\psi, \phi, \psi\}$ is satisfiable, which leads to contradiction.

Case 2: $\phi \notin \Gamma$

If $\phi \rightarrow \psi \notin \Gamma$ Then $\{\neg(\phi \rightarrow \psi), \neg\phi\}$ is satisfiable which can happen iff(Lecture-3)

$\{\phi \wedge \neg\psi, \neg\phi\}$ is satisfiable, which leads to contradiction.

$\phi \leftrightarrow \psi \in \Gamma$ **iff** ($\psi \in \Gamma$ **iff** $\phi \in \Gamma$)

Assume that $\phi \leftrightarrow \psi \in \Gamma$ holds. This happens iff $\phi \leftarrow \psi \in \Gamma$ and $\phi \rightarrow \psi \in \Gamma$. This happens iff $\psi \in \Gamma$ whenever $\phi \in \Gamma$ and $\phi \in \Gamma$ whenever $\psi \in \Gamma$ iff ($\psi \in \Gamma$ iff $\phi \in \Gamma$).

With this we have concluded showing all the cases of property 2. and proving Claim 2.

□

We now proceed to entertain the proof of Claim 3.

Proof. **Proof of Claim 3:**

Let Δ be a model set.This claim asserts that every model set is satisfiable. I.e., we need to show that there exists a model M_Δ such that $M_\Delta \vDash \Delta$. Since Δ is a complete set, this is the same as proving $M_\Delta \vDash \phi$ iff $\phi \in \Delta$. We will navigate to the claim through induction on the size of the formula.

Base Case(elaborated in Lec7): As has been mentioned in Lemma 1, all formulas in FOL are inductively defined by finitely applying quantifiers and connectives on $t_1 \equiv t_2$ and $P_i^n t_1 t_2 \dots t_n$ where all the P_i's come from a countable collection. We will treat primitive formulas as our base case and induct on the combinations of quantifiers and connectives.

We will prove the hypothesis for our base case in Lecture 7

Inductive Hypothesis: Assume the hypothesis holds for all formulas of size $\leq m$ where $m \in \mathbb{N}$. We would like to show that the hypothesis is fulfilled for all formulas of size $m + 1$, say ϕ

1.$\phi := \neg\psi$

2.$\phi := \chi \vee \psi$

3.$\phi := \chi \wedge \psi$

4.$\phi := \chi \rightarrow \psi$

5.$\phi := \chi \leftrightarrow \psi$

6.$\phi := \forall x\psi$

7.$\phi := \exists x\psi$

Inductive Step: 1.$M_\Delta \vDash \phi$ iff $M_\Delta \vDash \neg\psi$ iff $M_\Delta \nvDash \psi$ iff(inductive hypothesis) $\psi \notin \Delta$ iff(from the definition of a model set) $\neg\psi \in \Delta$ iff $\phi \in \Delta$

2.$M_\Delta \vDash \phi$ iff $M_\Delta \vDash \chi \vee \psi$ iff $M_\Delta \vDash \chi$ or $M_\Delta \vDash \psi$ iff(inductive hypothesis) $\phi \in \Delta$ or $\chi \in \Delta$ iff(from the definition of model set) $\phi \vee \chi \in \Delta$

3.$M_\Delta \vDash \phi$ iff $M_\Delta \vDash \chi \wedge \psi$ iff $M_\Delta \vDash \chi$ and $M_\Delta \vDash \psi$ iff(inductive hypothesis) $\phi \in \Delta$ and $\chi \in \Delta$ iff $\phi \wedge \chi \in \Delta$

4.$M_\Delta \vDash \phi$ iff $M_\Delta \vDash \psi \rightarrow \chi$ iff $M_\Delta \vDash \chi$ whenever $M_\Delta \vDash \psi$ iff(inductive hypothesis) $\chi \in \Delta$ whenever $\psi \in \Delta$ iff $\phi \rightarrow \chi \in \Delta$

5.$M_\Delta \vDash \phi$ iff $M_\Delta \vDash \psi \leftrightarrow \chi$ iff ($M_\Delta \vDash \chi$ iff $M_\Delta \vDash \psi$) iff(inductive hypothesis) ($\chi \in \Delta$ iff $\psi \in \Delta$) iff $\phi \leftrightarrow \chi \in \Delta$

We will show the remaining cases in Lecture 7.