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Lecture 9&10: Compactness Theorem – Part III

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Theorem 1 (Compactness Theorem) *Let Γ be an infinite set of formulas. Then Γ is satisfiable (sat) iff Γ is finitely satisfiable (fin-sat), meaning every finite subset of Γ is satisfiable.*

Quick Recap:

i. *On a drive to prove the (if part) theorem, we considered the following way: We will show, a fin-sat Γ is satisfiable by expanding it to some fin-sat and 'complete' set of formulas Δ (Claim 1) and thereby showing every such set of formulas is 'model set' (Claim 2) and every model set is satisfiable (Claim 3).*

ii. *Part-I discusses Claim 1 and 2. It further provides a validation of satisfiability on having a model μ_Δ.*

iii. *In Part-II, we have seen the definition of $\mu_\Delta = (\mathcal{D}_\Delta, \mathcal{I}_\Delta, \mathscr{G}_\Delta)$ and coming up with an equivalent model $\mu'_\Delta = (\mathcal{D}'_\Delta, \mathcal{I}'_\Delta, \mathscr{G}'_\Delta)$ in terms of segregating the terms of $\mathcal{D}_\Delta$ in some equivalent classes.*

Summary of this Part:
We need to show

$$\mu'_\Delta \models \forall x\psi \text{ iff } \forall x\psi \in \Delta.$$

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1 Proof of Compactness Theorem Part III

Before delving into the proof of our target, let us first go through a definition:

Definition 1.1 (Witness-Fulfilled set of Formulas) *A set of formulas Φ is said to be witness fulfilled if for every formula of the form $\exists x\phi \in \Phi$; $\phi[t/x] \in \Phi$ for some term t.*

Example. (Language of Arithmatic) Let P be any unary predicate symbol. Consider the set

$$\Phi = \{\exists x P(x), \neg P(0)\} \cup \{\neg P(S^k(0)) \mid k \geq 1\}$$

Because this Φ is finitely satisfiable, by compactness theorem Φ is itself satisfiable. □

Lemma 1.2 (Truth Lemma: Satisfiability under Quantifier Case)

$$\mu'_\Delta \models \forall x\psi \text{ iff } \forall x\psi \in \Delta.$$

To prove the above lemma, we would prove the following auxiliary propositions:

1. If any formula of form $\forall x\psi \in \Delta$, $\psi[t/x] \in \Delta$ for *all* term t.
2. If any formula of form $\exists x\psi \in \Delta$, $\psi[t/x] \in \Delta$ for *some* term t.

The need to use the propositions is – we will be using the validity $\models \forall x\psi \leftrightarrow \neg\exists x\neg\psi$ to prove the necessity and sufficiency of the Claim 1.2.

Proposition 1.3 *If any formula of form $\forall x\psi \in \Delta$, $\psi[t/x] \in \Delta$ for all terms t. That is*

$$\models \forall x\psi \rightarrow \psi[t/x].$$

Proof.

Let μ be a model such that $\mu \models \forall x\psi$.
$\Longrightarrow$ For any $d \in \mathcal{D}_\mu$ (domain of the model μ), by definition $\mu_{[x\to d]} \models \psi$.
$\Longrightarrow$ In particular, $\mu_{[x\to \mathscr{G}_\mu(t)]} \models \psi$ where t is substitutable for x in ψ.
$\Longrightarrow$ $\mu \models \psi[t/x]$.

And this holds true for any model μ satisfying $\forall x\psi$. Thus the proof. □

We will prove the second auxiliary proposition by proving the following proposition.

Proposition 1.4 *Any finitely satisfiable, complete set of formulas Γ can be extended to a finitely satisfiable, complete and witness-fulfilled set of formulas.*

Proof. To extend the language $\mathcal{L}$ i.e. the set of formulas Γ, let us introduce a countable set of new constant symbols, $\mathscr{D}$, where $\mathscr{D} = \{d_1, d_2, \ldots\}$. If $\mathcal{L} : (\mathcal{C}, \mathcal{F}, \mathcal{P})$ then the extended language $\mathcal{L}' : (\mathcal{C} \cup \mathscr{D}, \mathcal{F}, \mathcal{P})$. As $\mathcal{L}$ is countable, so is $\mathcal{L}'$. Thus, we enumerate the formulas in $\mathcal{L}'$ as $\beta_0, \beta_1, \beta_2, \ldots$.
Now, let us construct the sets of formulas Δ_0, Δ_1, $\Delta_2, \ldots$ as follows:

- $\Delta_0 = \Gamma$ (Γ is finitely satisfiable and complete by definition)
- If β_k is of form $\exists x\psi$, then

$$\Delta_{k+1} = \begin{cases} \Delta_k \cup \{\exists x\psi, \psi[d/x]\} & \text{if it is fin-sat, where } d \text{ is the least-indexed symbol in } \mathscr{D}, \\ & \quad \text{not occurring in } \Delta_k{}^{1} \\ \Delta_k & \text{otherwise.} \end{cases}$$

- Else,

$$\Delta_{k+1} = \begin{cases} \Delta_k \cup \{\beta_k\} & \text{if it is fin-sat} \\ \Delta_k & \text{otherwise.}, \end{cases}$$

[1] A constant symbol c not occurring in a formula ϕ can always be substitutable for x in ϕ.

By construction, note that every $\Delta_k \subseteq \Delta_{k+1}$ for $k \geq 0$.
We claim that the set of formulas $\Delta = \cup_{k\geq 0}\Delta_k$ is the one, we require. That is, Δ is finitely satisfiable, complete and witness-fulfilled. □

Lemma 1.5 (Generalisation of Constants) *If $\models \delta \to \psi[c/x]$, where c does not occur in δ and ψ, then $\models \delta \to \forall x\psi$.*

Proof. As usual, let us use contradiction again. Let $\mu \models \delta \to \forall x\psi$, by contradiction

$$\mu \models \delta \text{ and } \mu \not\models \forall x\psi$$
$$\text{thus } \mu \not\models \forall x\psi \text{ implies } \mu_{[x\to d]} \not\models \psi \text{ for some } d \text{ in the domain of the model } \mathcal{D}_\mu$$
$$\implies \mu_{[x\to d]} \models \neg\psi$$

To draw the contradiction, let us consider another model μ', identical to μ except for the fact that $\mathcal{I}_{\mu'}(c) = d$. Then, $\mu'_{[x\to d]} \models \neg\psi$, as ψ does not contain c.
Again, $\mu' \models \delta$; reason being δ is independent of c and μ satisfies δ. So, $\mu' \models \psi[c/x]$. So $\mu'_{[x\to \mathscr{G}'(c)]} \models \psi$ that is $\mu'_{[x\to d]} \models \psi$.
Thus a contradiction and hence $\delta \to \forall x\psi$ is validity. □

Proposition 1.6 *Let Δ be a finitely satisfiable and complete set of formulas. Let $\exists x\psi \in \Delta$ and c be a constant symbol not in Δ. Then, $\Delta \cup \{\psi[c/x]\}$ is finitely satisfiable.*

Proof. Let us prove using contradiction.

Assume that, $\Delta \cup \{\psi[c/x]\}$ is not finitely satisfiable.
$\implies \exists \Delta' \subseteq_{finite} \Delta$ and $\Delta' \cup \{\psi[c/x]\}$ is unsatisfiable.
$\implies \Delta' \models \neg\psi[c/x]$
$\implies \Delta' = \{\delta'_1, \delta'_2, \ldots, \delta'_n\} \models \neg\psi[c/x]$
$\implies \models (\delta'_1 \wedge \delta'_2 \wedge \ldots \wedge \delta'_n) \to \neg\psi[c/x]$. *HW* If $\Gamma = \{\gamma_i\} \models \phi$ then $\models (\wedge\gamma_i) \to \phi$
$\implies \models (\delta'_1 \wedge \delta'_2 \wedge \ldots \wedge \delta'_n) \to \forall x\neg\psi$. $\because c$ is not in Δ and Lemma 1.5

Because $\delta'_1 \wedge \delta'_2 \wedge \ldots \wedge \delta'_n \in \Delta$, we have $\forall x\neg\psi \in \Delta$ – a contradiction. Thus the proof of $\Delta \cup \{\psi[c/x]\}$ is finitely satisfiable. □

To make it a bit precise, prior to go into the proof of *Truth Lemma* let us analyse what we have done till this point. We introduced two auxiliary propositions – one is universal another is existential. The proof of the universal proposition (1.3) is straight forward. For proving the existential proposition we introduced the notion of witness fulfilledness. There we made a suitable construction that has been established using the Proposition 1.6 which in turn seek the help of Lemma 1.5.

Proof. (Proof of **Truth Lemma** 1.2)

Let us use contradiction again. That is

$$\begin{aligned}\forall x\psi \notin \Delta &\implies \neg\forall x\psi \in \Delta \implies \exists\neg\psi \in \Delta \\ &\implies \neg\psi[t/x] \in \Delta \text{ for some term } t\text{, by Proposition 1.3} \\ &\implies \psi[t/x] \notin \Delta\end{aligned}$$

Then by the induction hypothesis (from part II) $\mu'_\Delta \not\models \psi[t/x]$ meaning $\mu'_{\Delta_{[x\to\mathscr{G}'_\Delta(t)]}} \not\models \psi$ (that is $\mu'_{\Delta_{[x\to[t]]}} \not\models \psi$). So, $\mu'_\Delta \not\models \forall x\psi$. And we reached the contradiction. So the $\mu'_\Delta \models \forall x\psi \to \forall x\psi \in \Delta$ holds true.

If $\forall x\psi \in \Delta$, $\mu'_\Delta \models \forall x\psi$:

So, we need to show $\mu'_{\Delta_{[x\to[t]]}} \models \psi$, given $\forall x\psi \in \Delta$. Suppose not,

$$\mu'_{\Delta_{[x\to[t]]}} \not\models \psi \text{ for some term } t.$$

– If t is substitutable for x in ψ

We have $\mu'_\Delta \not\models \psi[t/x]$ meaning $\mu'_\Delta \models \neg\psi[t/x]$. Then by induction hypothesis, $\neg\psi[t/x] \in \Delta$. But by the given condition, $\models \forall x\psi \to \psi[t/x]$. thus a contradiction.

– If t is *not* substitutable for x in ψ. By Lemma 1.7

$\mu'_{\Delta_{[x\to[t]]}} \not\models \psi'$, where ψ' is an alphabetic variant satisfying Lemma 1.7.
So, $\mu'_\Delta \not\models \psi'[t/x]$ $\because$ Construction of ψ allows so.
So, $\psi'[t/x] \notin \Delta$ (by induction hypothesis).
So, $\forall x\psi' \notin \Delta$ (since $\models \forall x\phi \to \phi[t/x]$ where t is substitutable for x in ϕ).
So, $\forall x\psi \notin \Delta$ which is a contradiction.

Note that, we have used the following implication here:

$$^{HW} \models \phi \leftrightarrow \phi' \text{ implies } \models \forall x\phi \leftrightarrow \forall x\phi'$$

This completes the proof. □

Lemma 1.7 ^{HW} *Given any formula ϕ, a variable x and term t; there exists a formula ϕ' (an alphabetic variant of ϕ) obtained by renaming the bound variables in ϕ such that t is substitutable for x in ϕ' and $\models \phi \leftrightarrow \phi'$.*

Looking Back and Recap:

1. Start with a finitely satisfiable set Γ.
2. Extend the set to form a finitely satisfiable and complete set.
3. Show that the extended set becomes a model set.

4. Then show that this set has a model. This gave the proof of the truth lemma for quantifier-free formulas (zeroth order logic).

5. Extend the finitely satisfiable and complete set of formulas to a finitely satisfiable, complete and witness-fulfilled set to get the proof of the truth lemma for the quantified formulas.

While doing all these, at some point, we extended the language from $\mathcal{L}$ to $\mathcal{L}'$. Thus the model we constructed is a model for Γ in the language $\mathcal{L}'$. But we need a model for Γ in the language $\mathcal{L}$. Here, $\mathcal{L} : (\mathcal{C}, \mathcal{F}, \mathcal{P})$ and $\mathcal{L}' : (\mathcal{C} \cup \mathscr{D}, \mathcal{F}, \mathcal{P})$

How to get the required model?

Let μ' be a model for the language $\mathcal{L}'$, that is $\mu' = (\mathcal{D}', \mathcal{I}', \mathscr{G}')$, where $\mathcal{I}'$ is defined for $\mathcal{C} \cup \mathscr{D}, \mathcal{F}$ and $\mathcal{P}$. Now, we construct a restricted model, $\mu'|_{\mathcal{L}}$ as follows: $\mu = (\mathcal{D}, \mathcal{I}, \mathscr{G})$, where $\mathcal{D} = \mathcal{D}'$, $\mathscr{G} = \mathscr{G}'$ and $\mathcal{I}$ is defined for $\mathcal{C}, \mathcal{F}$ and $\mathcal{P}$.

Now, Γ is a set of formulas in $\mathcal{L}$, so it does not contain any constant symbol from $\mathscr{D}$. In the proof earlier, we constructed a model μ'_Δ for Γ in the language $\mathcal{L}'$. However, as no symbol from $\mathscr{D}$ occur in Γ, $\mu'_\Delta|_{\mathcal{L}}$ forms a model for Γ as well. And, $\mu'_\Delta|_{\mathcal{L}}$ is a model with respect to the language $\mathcal{L}$. Hence, Γ is satisfiable, that is, every finitely satisfiable set of formulas is satisfiable. This completes the proof of *Compactness Theorem*.

HW Proof of Lemma 1.7

We construct ϕ' by recursion on ϕ:

- If ϕ is atomic, $\phi' = \phi$.
- $(\neg\phi)' = (\neg\phi')$ and $(\phi \to \psi)' = (\phi' \to \psi')$
- $(\forall y \phi)' = \forall z(\phi')[y/z]$, where z does not appear in ϕ' or t or x.

Showing the validity is straight forward.

HW If $\Gamma = \{\gamma_1, \ldots, \gamma_n\} \models \phi$ then $\models (\gamma_1 \wedge \cdots \wedge \gamma_n) \to \phi$.

To prove this, lets get back to the definition of *Consequence Relation*. Say $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ then for all models μ, $\mu \models \gamma$ for every $\gamma \in \Gamma$ imply $\mu \models \phi$. So for all model μ, $\mu \models \gamma$ for every $\gamma \in \Gamma$ implies $\mu \models (\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n)$ too. So, $(\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n) \to \phi$ is valid.