Mathematical Logic
Homework 2

1. Let $\mathcal{L}$ denote the language of classical propositional logic. Let $\Gamma \subseteq \mathcal{L}$ and $\alpha, \beta \in \mathcal{L}$. Show that $\Gamma \cup \{\alpha\} \vdash \beta$ if and only if $\Gamma \vdash \alpha \to \beta$.

2. Show that the following formulas are tautologies of classical propositional logic:

   (a) $(\alpha \wedge \beta) \leftrightarrow \neg(\neg\alpha \vee \neg\beta)$

   (b) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \wedge \beta) \to \gamma)$

3. Show that the following formulas are theorems of classical propositional logic:

   (a) $\neg\neg\alpha \leftrightarrow \alpha$

   (b) $(\neg\alpha \to \alpha) \to \alpha$

4. Let us consider the binary number system. When we add two numbers of at most two digits, say ab and cd, we get a number of at most three digits, say pqr. For example, $11 + 10 = 101$. Using standard connectives, write formulas that express p, q, and r as functions of a, b, c, and d.

5. Let $G = (V, E)$ be a graph with a finite non-empty set of vertices, without any self-loops. Let k be a non-zero natural number. For each $x \in V$ and $1 \leq i \leq k$, let $p_{x,i}$ denote a propositional variable. Define a set of formulas, F say, on the set of variables $p_{x,i}$, such that F is satisfiable if and only if G is k-colourable.

6. Consider the following clues to a treasure hunt:

   (a) If this house is next to a lake, then the treasure is not in the kitchen.

   (b) If the tree in the front yard is an elm, then the treasure is in the kitchen.

   (c) The house is next to a lake.

   (d) The tree in the front yard is an elm or the treasure is buried under the flagpole.

   Where is the treasure? Use semantic consequence relation of Classical Propositional Logic to justify your answer.

7. Find a valuation $V : \{p, q, r\} \to \{0, 1\}$ for each of the following formulas so that the corresponding truth conditions are satisfied:

   (a) $V((p \to (q \to r)) \to ((p \wedge q) \to r)) = 1$

   (b) $V(\neg(\neg p \wedge (p \to q)) \to \neg q) = 1$

   (c) $V(((p \vee q) \to r) \vee ((r \vee q) \to p)) = 0$

8. Let M be a maximal consistent set of classical propositional logic formulas. Show that the following holds:

   (a) If $\vdash_{CPL} \alpha$ then $\alpha \in \mathsf{M}$

   (b) $\alpha \vee \beta \in \mathsf{M}$ if and only if ($\alpha \in \mathsf{M}$ or $\beta \in \mathsf{M}$)

9. Consider the following simple 3×3 board to be filled up with the numbers 1, 2 and 3 so that there is no repetition across rows or columns. Fill up the board according to these rules and formalize the reasoning process in a logic.

$$\begin{array}{ccc} 1 & \cdot & \cdot \\ \cdot & \cdot & 2 \\ \cdot & \cdot & \cdot \end{array}$$

10. If $C \subseteq \{\neg, \wedge, \vee, \to, \perp\}$ is an adequate set of connectives, then show that $\neg \in C$ or $\perp \in C$. Is $\{\neg, \leftrightarrow\}$ an adequate set of connectives ? Justify your answer.