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Explaining Lowenheim - Skolem (Tarski) Theorem:

A general version: (a) Let Γ be a satisfiable set of formulas in a language of cardinality λ. Then Γ is satisfiable in a model of cardinality $\leq \lambda$.

(b) Let Γ be a satisfiable set of formulas in a language of cardinality λ such that Γ has an infinite model. Then, for every cardinality $\kappa \geq \lambda$, Γ has a model of cardinality κ.

A corollary: Let Γ be a satisfiable set of formulas having an infinite model. Then Γ has a countably infinite model.

Proving DLS Theorem via Compactness Theorem

Proved the part (a) for the countable language and did not mention 'finite'.

Question: Can the model constructed via CT be finite?

Proving Los - Vaught Test

Missed the very important assumption that T does not have any finite models, while using the corollary.

Proof systems for FOL.

We discussed three proof systems for CPL: Hilbert's axiomatization, Natural Deduction and Sequent Calculus. Let us do the same for FOL. We have already discussed Hilbert's axiomatization and proved completeness for FOL. Now, it's time to discuss the other two:

Natural Deduction.

$\wedge$-I $\dfrac{\alpha \quad \beta}{\alpha \wedge \beta}$ $\wedge$-E_1 $\dfrac{\alpha \wedge \beta}{\alpha}$ $\wedge$-E_2 $\dfrac{\alpha \wedge \beta}{\beta}$

$\vee$-I_1 $\dfrac{\alpha}{\alpha \vee \beta}$ $\vee$-I_2 $\dfrac{\beta}{\alpha \vee \beta}$ $\vee$-E $\dfrac{\alpha \vee \beta \quad \begin{matrix}[\alpha] \\ \vdots \\ \gamma\end{matrix} \quad \begin{matrix}[\beta] \\ \vdots \\ \gamma\end{matrix}}{\gamma}$

$\to$-I

$$\dfrac{\begin{matrix}[\alpha] \\ \vdots \\ \beta\end{matrix}}{\alpha \to \beta}$$

discharging α

$\to$-E

$$\dfrac{\alpha \qquad \alpha \to \beta}{\beta}$$

$\neg$-I

$$\dfrac{\begin{matrix}[\alpha] \\ \vdots \\ \bot\end{matrix}}{\neg\alpha}$$

$\neg$-E

$$\dfrac{\begin{matrix}[\neg\alpha] \\ \vdots \\ \bot\end{matrix}}{\alpha}$$

Others : $\dfrac{\neg\neg\alpha}{\alpha}$; $\dfrac{\alpha \qquad \neg\alpha}{\bot}$; $\dfrac{\bot}{\beta}$

$\neg\neg$-rule $\qquad$ $\bot$-I $\qquad$ $\bot$-E

What are the rules for $\forall$ and $\exists$?

$\forall$-I

$$\dfrac{\varphi[c/x]}{\forall x \varphi.}$$

, c does not occur in φ or, in any undischarged assumption.

$\forall$-E

$$\dfrac{\forall x \varphi}{\varphi[t/x]}$$

, t is substitutable for x in φ.

$$\dfrac{\forall x \varphi}{\varphi[c/x]}$$

: Another version.

$\exists$-I

$$\dfrac{\varphi[c/x]}{\exists x\,\varphi}$$

$\exists$-E

$$\dfrac{\exists x\,\varphi \qquad \begin{matrix}\varphi[c/x]\\ \vdots \\ \beta\end{matrix}}{\beta},$$

c does not occur in φ or, in any undischarged assumption.

Examples :

1. Derive $\forall x \neg \forall y \neg P(x,y)$ from $\forall x \neg \forall y\,(P(x,y) \to Q(x,y))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P(c,d)] \qquad \dfrac{[\forall y \neg P(c,y)]}{\neg P(c,d)}\ \forall\text{-E}}{Q(c,d)}\ \bot\text{-rules}}{P(c,d) \to Q(c,d)}\ \to\text{-E}}{\forall y\,(P(c,y) \to Q(c,y))}\ \forall\text{-I} \qquad \dfrac{\overset{\text{Assumption}}{\forall x \neg \forall y\,(P(x,y) \to Q(x,y))}}{\neg \forall y\,(P(c,y) \to Q(c,y))}\ \forall\text{-E}}{\neg \forall y \neg P(c,y)}\ \neg\text{-I}}{\forall x \neg \forall y \neg P(x,y)}\ \forall\text{-I}$$

2. Derive $\exists x\,(P(x) \wedge \neg R(x))$ from
$\exists x\,(P(x) \wedge Q(x))$ and $\neg \exists x\,(Q(x) \wedge R(x))$

$$
\dfrac{
\overset{\text{Assumption}}{\exists x\,(P(x)\wedge Q(x))}
\qquad
\dfrac{
\dfrac{
\wedge\text{-}E_1\;\dfrac{[P(a)\wedge Q(a)]}{P(a)}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{[P(a)\wedge Q(a)]}{Q(a)}\;\wedge\text{-}E_2
\qquad [R(a)]
}{Q(a)\wedge R(a)}\;\wedge\text{-}I
}{\exists x\,(Q(x)\wedge R(x))}\;\exists\text{-}I
\qquad
\overset{\text{Ass.}}{\neg\exists x\,(Q_x\wedge R(x))}
}{\neg R(a)}\;\neg\text{-}I
}{P(a)\wedge\neg R(a)}\;\wedge\text{-}I
}{\exists x\, P(x)\wedge\neg R(x)}\;\exists\text{-}I
}{\exists x\,(P(x)\wedge\neg R(x))}\;\exists\text{-}E
$$

An application of :

$$
\dfrac{\exists x\;\alpha \qquad \begin{matrix}\alpha[c/x]\\ \vdots\\ \beta\end{matrix}}{\beta}
$$

Sequent Calculus

We only discuss the quantifier rules here.

$\forall L$ $$\frac{\Gamma, \alpha[t/x] \vdash \Delta}{\Gamma, \forall x \alpha \vdash \Delta}$$

$\forall R$ $$\frac{\Gamma \vdash \alpha[y/x], \Delta}{\Gamma \vdash \forall x \alpha, \Delta},$$

y is not free in the conclusion

$\exists L$ $$\frac{\Gamma, \alpha[y/x] \vdash \Delta}{\Gamma, \exists x \alpha \vdash \Delta},$$

where y is not free in the conclusion

$\exists R$ $$\frac{\Gamma \vdash \alpha[t/x], \Delta}{\Gamma \vdash \exists x \alpha, \Delta}$$

Examples:

1.

$$\cfrac{\cfrac{\cfrac{}{P(f(y)) \vdash P(f(y))}\ Ax}{\forall x P(x) \vdash P(f(y))}\ \forall\text{-}L}{\forall x P(x) \vdash \forall y P(f(y))}\ \forall\text{-}R$$

2.

$$\dfrac{}{P(x) \vdash P(x)}\ \text{Ax}$$

$$\dfrac{P(x) \vdash P(x)}{P(x), Q(x) \vdash P(x)}\ \text{W-L}$$

$$\dfrac{P(x), Q(x) \vdash P(x)}{(P(x) \wedge Q(x)) \vdash P(x)}\ \wedge\text{-L}$$

$$\dfrac{(P(x) \wedge Q(x)) \vdash P(x)}{\forall x (P(x) \wedge Q(x)) \vdash P(x)}\ \forall\text{-L}$$

$$\dfrac{\forall x (P(x) \wedge Q(x)) \vdash P(x)}{\forall x (P(x) \wedge Q(x)) \vdash \forall x\, P(x)}\ \forall\text{-R}$$

3 -

$$\text{W-R}\ \dfrac{\dfrac{}{\alpha \vdash \alpha}\ \text{Ax}}{\alpha \vdash \alpha, \beta} \qquad \dfrac{\dfrac{}{\beta \vdash \beta}\ \text{Ax}}{\alpha, \beta \vdash \beta}\ \text{W-L}$$

$$\dfrac{\alpha \vdash \alpha, \beta \qquad \alpha, \beta \vdash \beta}{\alpha, \alpha \to \beta \vdash \beta}\ \to\text{-L}$$

$$\dfrac{\alpha, \alpha \to \beta \vdash \beta}{\alpha, \forall x (\alpha \to \beta) \vdash \beta}\ \forall\text{-L}$$

$$\dfrac{\alpha, \forall x (\alpha \to \beta) \vdash \beta}{\alpha, \forall x (\alpha \to \beta) \vdash \forall x\, \beta}\ \forall\text{-R} \quad (x \text{ is not free in } \alpha)$$

$$\dfrac{\alpha, \forall x (\alpha \to \beta) \vdash \forall x\, \beta}{\forall x (\alpha \to \beta) \vdash \alpha \to \forall x\, \beta}\ \to\text{-R}$$

$$\dfrac{\forall x (\alpha \to \beta) \vdash \alpha \to \forall x\, \beta}{\vdash (\forall x (\alpha \to \beta)) \to (\alpha \to \forall x\, \beta)}\ \to\text{R}$$

x does not occur free in α.

4.

$$
\vee\text{-}L\;\dfrac{\exists\text{-}L\;\dfrac{\exists\text{-}R\;\dfrac{\vee\text{-}R\;\dfrac{W\text{-}R\;\dfrac{Ax\;\dfrac{}{P(x) \vdash P(x)}}{P(x) \vdash P(x), Q(x)}}{P(x) \vdash P(x) \vee Q(x)}}{P(x) \vdash \exists x\,(P(x) \vee Q(x))}}{\exists x\, P(x) \vdash \exists x\,(P(x) \vee Q(x))} \qquad \overset{\text{Similar}}{\exists x\, Q(x) \vdash \exists x\,(P(x) \vee Q(x))}}{\exists x\, P(x) \vee \exists x\, Q(x) \vdash \exists x\,(P(x) \vee Q(x))}
$$

In the next class we will continue with the study of proof systems in CPL and FOL.