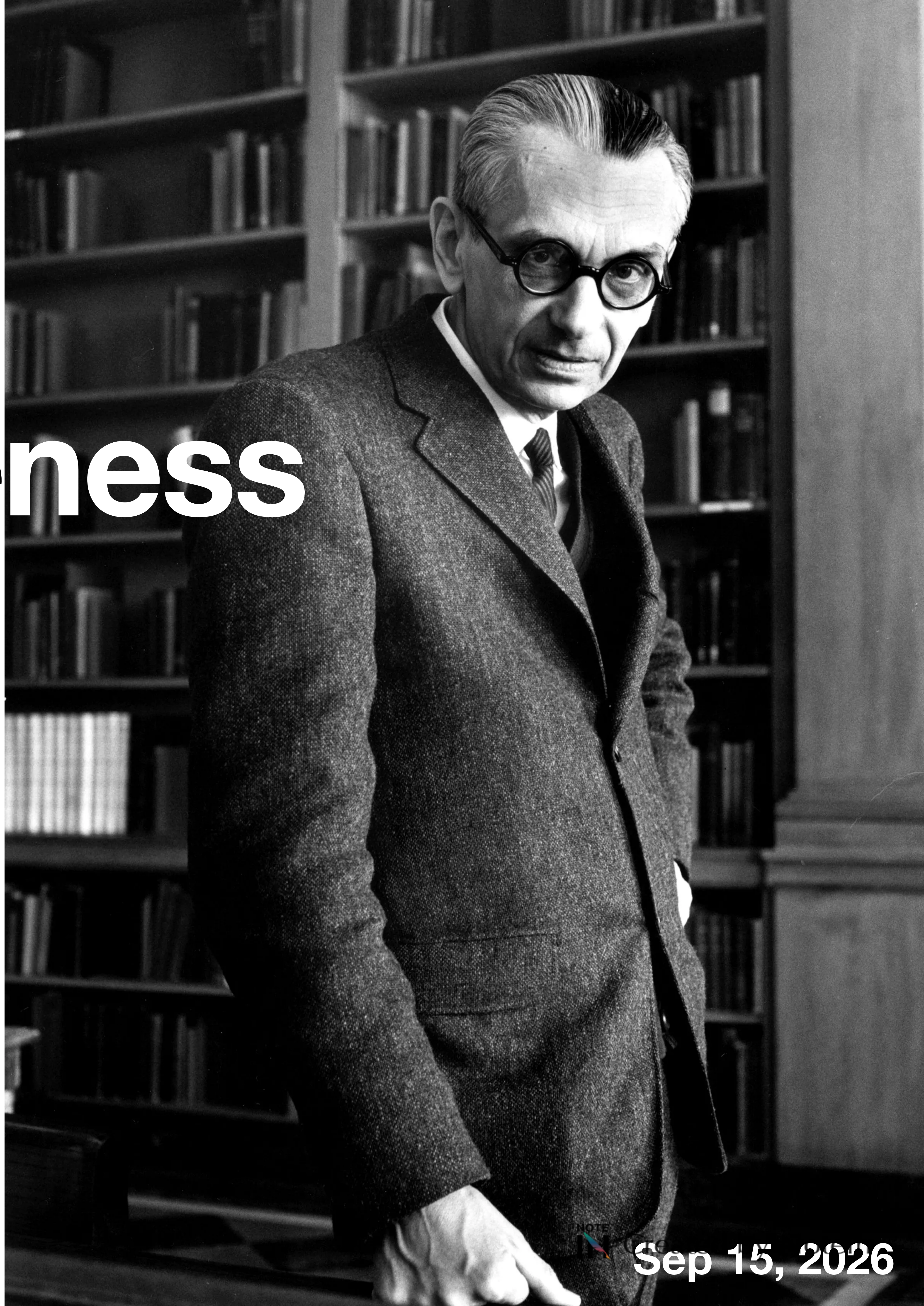
Gödel’s Incompleteness Theorems

A formal study of introspection



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Syntax and Semantics

Fundamental idea in Logic

- Syntax is an abstract symbol. Semantics is the meaning we are attaching to syntax.
- Number vs Numeral
- Programming languages vs their natural language understanding.
- A nice quote by Boolos, from Logic of Provabiliy:

> "Caution! (e.g.) ⊥ is a symbol. What sort of thing the symbol ⊥ is, whether it is a shape with a stem and a base, or whether it is a set or a number, or something else entirely, we have not said (and need not say). '⊥', in contrast, is the name of ⊥ (and is itself also a symbol, maybe the same one as ⊥, maybe not); '⊥' definitely does have a stem and a base."

Peano Arithmetic

'God made the natural numbers, all else is the work of man' —Kronecker

- Language: $\mathscr{L} = (+ , \ \times , S, 0)$

- Structure: The usual set of natural number $\mathbb{N}$, where $+$ and $\times$ have usual meaning. $S : \mathbb{N} \to \mathbb{N}$ is the successor function. About notation: $\overline{(\cdot)}$ refers to an abbreviation of a term in $\mathscr{L}$. For example: $\overline{2} = SS0$.

- Example sentence:

$$\psi_G := \forall e \left((Even(e) \wedge \exists v(SS0 + Sv = e)) \to \exists n_1 \exists n_2 (Prime(n_1) \wedge Prime(n_2) \wedge n_1 + n_2 = e) \right)$$

↳ Goldbach conjecture.

$\varphi(d, x) := \exists q(q \times d) = x$

$Prime(p) := \neg(p = S\ 0) \wedge \forall d(\varphi(d, p) \to (d = S\ 0 \vee d = p))$

$Even(e) := \varphi(SS0, e)$

Peano Arithmetic

‘God made the natural numbers, all else is the work of man’ —Kronecker

1. $\forall x(0 \neq Sx)$
2. $\forall x \forall y(Sx = Sy \rightarrow x = y)$
3. $\forall x(x + 0 = x)$
4. $\forall x \forall y(x + Sy = S(x + y))$
5. $\forall x(x \times 0 = 0)$
6. $x \times Sy = (x \times y) + x$

Robinson Arithmetic

(Q: $\underbrace{\forall x(\neg(x = 0) \rightarrow \exists y(x = Sy))}_{\text{not provable in Q}}$)

Another example would be : $Q \nvdash \forall x\,(0 + x) = x$
but, for all $n \in \mathbb{N}$, $Q \vdash 0 + \bar{n} = \bar{n}$. Why?
hint: Induction Scheme availability.

7. Induction Schema: $\underline{\varphi(0) \rightarrow \forall x(\varphi(x) \rightarrow \varphi(Sx)) \rightarrow \forall x \varphi}$

First Order Logic of

Peano Arithmetic

1. All the propositional axioms mentioned earlier.

2. $\forall x(\varphi \to \psi) \to \forall x\varphi \to \forall x\psi$

3. $\forall x\varphi \to \varphi_x(t)$ t is substitution for x in φ $\varphi_x(t)$ is same as $\varphi[t/x]$

4. $\forall x(x = x)$

5. $x = y \to t = t'$ where t' is obtained from t by replacing one occurrence of x by y

6. $x = y \to (\varphi \to \varphi')$ where φ is atomic and φ' is obtained from φ by replacing one occurrence of x by y

$$\frac{\varphi \quad \varphi \to \psi}{\psi} \text{(MP)}$$

$$\frac{\varphi}{\forall x\varphi} \text{(Gen)} \quad \text{Where } x \text{ is not free in any active premise}$$

First Order Logic of

Peano Arithmetic: What is a proof?

A proof or derivation of φ in PA, denoted as $PA \vdash \varphi$ is a finite sequence:

$$\vdash \varphi_0$$

$$\vdash \varphi_1$$

$$\vdots$$

$$\vdash \varphi_n$$

Such that each element is either an instance of the axioms in PA or an instance of the logical axioms or is a result of Modus Ponens of earlier elements in the sequence or an application of an Generalization of earlier statement and the last element φ_n is φ, the formula that is to be derived.

First Order Logic of

Peano Arithmetic: Example proof of $\forall y \forall x(Sx + y = S(x + y))$

1. $PA \vdash \forall x(Sx + 0 = Sx)$ by PA3$(\varphi(0))$

2. $PA \vdash Sx + y = S(x + y) \rightarrow Sx + Sy = S(Sx + y) = SS(x + y) = S(x + Sy)$
   by PA4, antecedent(?).

3. $PA \vdash \forall y \forall x(Sx + y = S(x + y) \rightarrow Sx + Sy = S(x + Sy))$
   $(\forall x(\varphi(x) \rightarrow \varphi(Sx)))$

4. $PA \vdash \forall x \forall y(Sx + y = S(x + y))$

First Order Logic of

Peano Arithmetic: The notion of Provability

How does one do this? Well we can <u>encode</u> using natural numbers, based on an agreed upon rule.

- Suppose we have a relation $Prf(x, p)$, which takes in a proof x and tells us whether it proves the statement p and it can be represented inside PA.

$$Prf(\overline{x_0}, \ulcorner \varphi \urcorner)$$

holds true if $\overline{x}_0$ is the encoding of the proof of the statement φ.

- Let, $Prov(x) := \exists y Prf(y, x)$ and $U(p) := \neg Prov(diag(p))$. Sometimes $Bew(x)$ from 'Beweis'

- Where $diag(\ulcorner \varphi \urcorner) = \ulcorner \exists (y = \ulcorner \varphi \urcorner \wedge \varphi) \urcorner$. Now please focus on this

$$\mathfrak{G} := \exists y (y = \ulcorner U \urcorner \wedge U(y))$$

essentially it's saying $\mathfrak{G}$ is equivalent to $U(\ulcorner U \urcorner)$ which is essentially, $\neg Prov(\ulcorner U(\ulcorner U \urcorner) \urcorner)$, but that is $\neg Prov(\mathfrak{G})$

$\therefore\ G \sim \neg Prov(\ulcorner G \urcorner)$

Gödel's First Incompleteness

In a rather incomplete fashion…

- If PA is sound then $PA \nvdash \mathfrak{G}$ and $PA \nvdash \neg\mathfrak{G}$. How?
- Notice $\mathbb{N} \models \mathfrak{G}$ iff it is not provable in PA
- However, what about the extensive discussion regarding "purely symbolic manipulation"?
- Was this argument of symbolic nature?

Proof: Assume PA is sound.

- Say $PA \vdash G$. Then $\mathbb{N} \models G$. But that means G is not provable in PA. Therefore a contradiction!
- Say $PA \vdash \neg G$. Then $\mathbb{N} \models \neg G$. But that will imply G is provable in PA. But that makes the system inconsistent. Hence a contradiction ∎

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To be or ¬ be

- Inconsistency(of a theory): There is a formula $\varphi \in \mathscr{L}$, such that $PA \vdash \varphi$ and $PA \vdash \neg\varphi$.
- Consistency: not Inconsistent.
- $\omega-$inconsistency: there is $\varphi(x) \in \mathscr{L}$, such that $PA \vdash \exists x\ \varphi(x)$ and for every $m \in \mathbb{N}$, $PA \vdash \neg\varphi(m)$
- $\omega-$consistency : not $\omega-$inconsistent.
- Fact: There are consistent but not $\omega-$consistent theories of arithmetic. (Consider $PA + \neg\mathfrak{G}$)

Gödel's First Incompleteness

Still in an incomplete and offensive fashion…

If PA is consistent then $PA \nvdash \mathfrak{G}$

If PA is $\omega-$consistent then $PA \nvdash \neg \mathfrak{G}$

Books to be followed:

(1) Peter Smith : An Introduction to Gödel's Theorem