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Gödel's Incompleteness Theorems

A formal study of introspection

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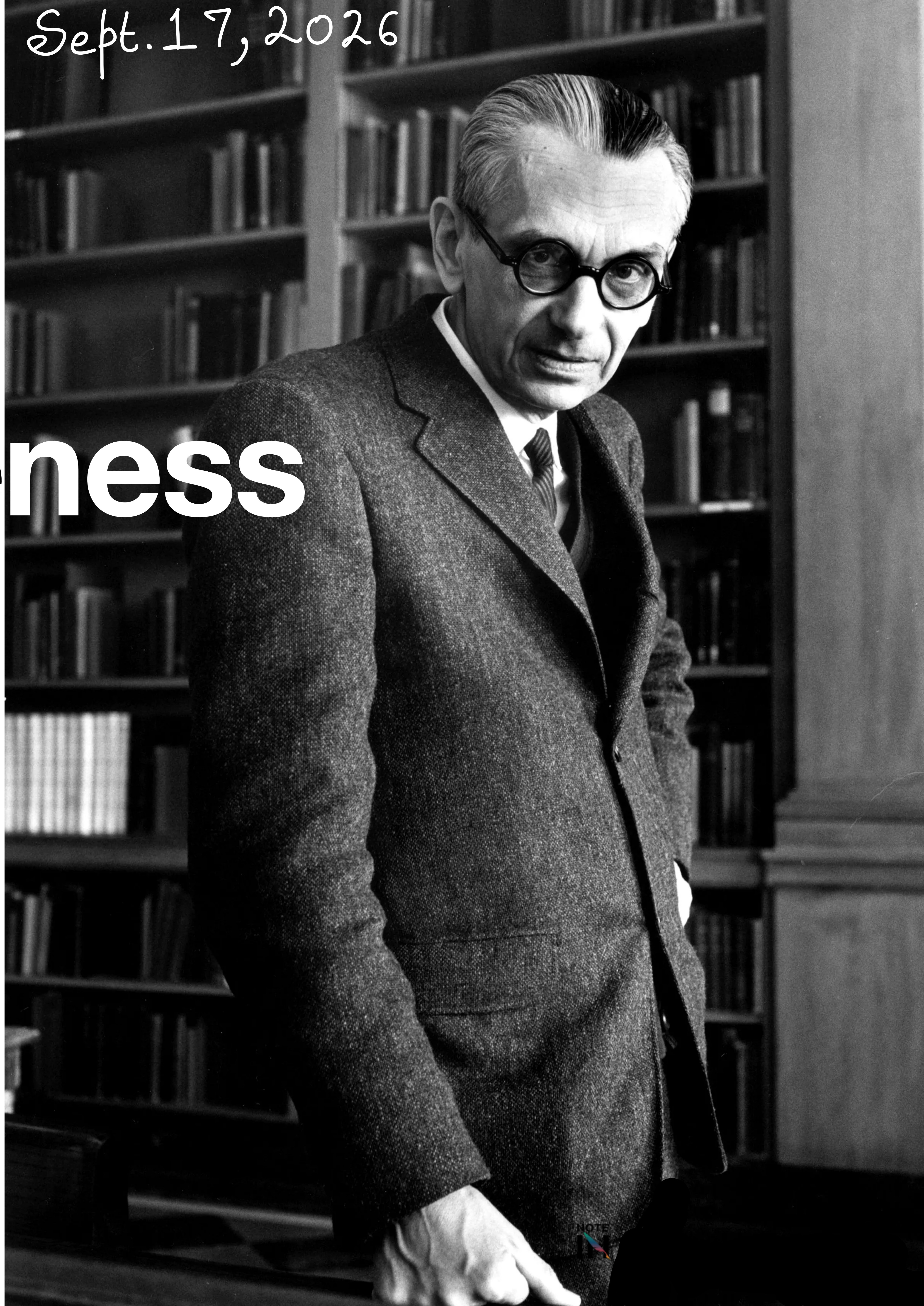

Pulling the meta-theory inside formal theory

Some tools…

Prime numbers:2, 3, 5, 7, 11, 13, 17
, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59,
61, 67, 71, 73, 79, 83, 89, 97, 101, 103,
107, 109, 113, 127, 131, 137, 139…

- Symbols we have:

$$\bot\,,\ \to\,,\ \forall,\ =\,,\ 0,\ S,\ +\,,\ \times\,,\ (,\ )$$

- We can put Gödel numbering as $\overline{1},\ \overline{3},\ \overline{5},\ \overline{7},\ \overline{9},\ \overline{11},\ \overline{13},\ \overline{15},\ \overline{17}$ and $\overline{19}$. Rest of the Odd numbers are for variables. Something like this: $\ulcorner v_i \urcorner := 2 \times i + 21$

- Then for a formula $\ulcorner \forall v_0(v_1 = S00 \times v_0 \to \bot\,) \to \bot\,\urcorner =$

$$2^{\ulcorner\forall\urcorner} \times 3^{\ulcorner v_0\urcorner} \times 5^{\ulcorner(\urcorner} \times 7^{\ulcorner v_1\urcorner} \times 11^{\ulcorner=\urcorner} \times 13^{\ulcorner S\urcorner} \times 17^{\ulcorner 0\urcorner} \times 19^{\ulcorner 0\urcorner} \times 23^{\ulcorner v_0\urcorner} \times 29^{\ulcorner\to\urcorner} \times 31^{\ulcorner\bot\urcorner} \times 37^{\ulcorner)\urcorner} \times 41^{\ulcorner\to\urcorner} \times 43^{\ulcorner\bot\urcorner}$$

$$= 2^5 \times 3^{21} \times 5^{17} \times 7^{23} \times 11^7 \times 13^{11} \times 17^9 \times 19^9 \times 23^{21} \times 29^3 \times 31^1 \times 37^{19} \times 41^3 \times 43^1$$

Revisiting to p.r. functions

For loops in disguise… (P.pp-108) (ch-14)

- A relation $P(\bar{x})$ is a p.r. relation, if the characteristic function is a p.r. function.

- If $f(\bar{x})$ is an n-place p.r. function, then the corresponding relation expressed by $f(\bar{x}) = y$ is an $n + 1$-place p.r. relation.

- Any truth-functional combination of p.r. relations is p.r.

- Any relation defined from a p.r. property or relation by bounded quantifications is also p.r.

- If P is a p.r. relation, then the function $f(n) = (\mu x \leq n)P(x)$ is p.r And generalizing, suppose that $g(n)$ is a p.r. function; then $f'(n) = (\mu x \leq g(n))P(x)$ is also p.r.

- Any function defined by cases from other p.r. functions is also p.r.

Revisiting to p.r. functions (ch-14)

For loops in disguise... (P.pp-111)

- The relations $m = n$, $m < n$ and $m \leq n$ are primitive recursive. First define monus in p.r. Then use it to define modulus. Also define signum function.

- The relation $m | n$ that holds when m is divides n is primitive recursive.
  $\exists y < m [n = m \times y \wedge 0 < m \wedge 0 < y]$

- Let Prime(n) be true just when n is a prime number. Then P$(\cdot)$ is a p.r. property. Try it yourself

- List the primes as $\pi_0, \pi_1, \pi_2, \ldots$ Then the function $\pi(n)$ whose value is π_n is p.r.
  Use the fact that any prime greater than k is bounded by k! + 1

- Let exf(n, i) be the possibly zero exponent of π_i — the $i + 1$-th prime — in the factorization of n. Then exf$(\cdot)$ is a p.r. function.

- Let len(0) = len$(1) = 0$; and when $n > 1$, let len(n) be the 'length' of n's factorization, i.e. the number of distinct prime factors of n. Then len$(\cdot)$ is again a p.r. function.

Revisiting to p.r. functions

For loops in disguise…

- $\mathrm{exf}(n, i) := (\mu x \leq n)\left(\pi_i^{x} \mid n \wedge \neg\, \pi_i^{x+1} \mid n\right)$

- $\mathrm{len}(n) := \sum_{i=0}^{n} \mathrm{pd}(i, n)$, where $\mathrm{pd}(i, n) := \mathrm{Prime}(i) \wedge i \mid n$

Try to write it out formally.

Encoding using p.r.

Our ‘glue’ is p.r.

Ch-19, 20

- There is a two-place concatenation function, standardly represented by $m \circ n$, such that (i) if m and n are the g.n. of two expressions then $m \circ n$ is the g.n. of the result of writing the first expression followed by the second, and (ii) this function is primitive recursive.

Example: Let $\ulcorner \forall v_1 \urcorner = m = 2^5 \times 3^{23}$ and
$\ulcorner (v_1 = 0 \to \perp) \urcorner = n = 2^{17} \times 3^{23} \times 5^7 \times 7^9 \times 11^3 \times 13^1 \times 17^{19}$, then

$$\ulcorner \forall v_1 (v_1 = 0 \to \perp) \urcorner = m \circ n = 2^5 \times 3^{23} \times 5^{17} \times 7^{23} \times 11^7 \times 13^9 \times 17^3 \times 19^1 \times 23^{19}$$

This concatenation function $\circ(\cdot, \cdot)$ can be written as: ($\circ$ forms a monoid on $\mathbb{N}$, with 0 being identity)

$$m \circ n = (\mu x \leq \pi_{m+n}^{m+n})[(\forall i < \mathrm{len}(m))\{\mathrm{exf}(x,i) = \mathrm{exf}(m,i)\} \wedge$$
$$(\forall i < \mathrm{len}(n))\{\mathrm{exf}(x, i + \mathrm{len}(m)) = \mathrm{exf}(n,i)\}]$$

Why?

Encoding using p.r.

Ch-19, 20

Genesis of a term:

- The function num : $\{0\} \cup \{S^n 0 \mid n > 0\} \rightarrow \{\ulcorner \varphi \urcorner \mid \varphi \in \mathscr{L}_{PA}\}$ is p.r:
  $\text{num}(0) = \ulcorner 0 \urcorner$
  $\text{num}(Sx) = \ulcorner S \urcorner \circ \text{num}(x)$

- The function $Diag : \mathscr{L}_{PA} \rightarrow \mathscr{L}_{PA}; \varphi \mapsto \exists v_1(v_1 = \ulcorner \varphi \urcorner \wedge \varphi)$ can be turned to a function on Gödel encodings: (There are huge elephants in the room lurking here.Can you see?)
  $\text{diag}(n) = \ulcorner \exists v_1(v_1 = \urcorner \circ \text{num}(0) \circ \ulcorner \wedge \urcorner \circ n \circ \ulcorner) \urcorner$

- $\text{Var}(n) = \exists x \leq n \ (n = 2 \times x + 17)$: For checking if n is a encoding of a variable

- $\text{TermSeq}(n) := \forall k < \text{len}(n)\{\text{exf}(n,k) = \ulcorner 0 \urcorner \vee \text{Var}(exf(n,k)) \vee$
  $\exists j < k \ (\text{exf}(n,k) = \ulcorner \text{S} \urcorner) \circ \text{exf}(n,j)) \vee$
  $\exists i,j < k \ (\text{exf}(n,k) = \ulcorner (\urcorner \circ \text{exf}(n,i) \circ \ulcorner + \urcorner \circ \text{exf}(n,j) \circ \ulcorner) \urcorner) \vee$
  $\exists i,j < k \ (\text{exf}(n,k) = \ulcorner (\urcorner \circ \text{exf}(n,i) \circ \ulcorner \times \urcorner \circ \text{exf}(n,j) \circ \ulcorner) \urcorner)\}$

Encoding using p.r.

Genesis of a term:

Ch-19, 20

- $\text{Term}_{n-p.r.}(n) = \exists x\ (\text{TermsSq}(x) \wedge n = exf(x, \text{len}(x) - 1))$

- But can this be captured by p.r. ?

- What do you think?

TermSequence : Say we want to encode $(S^2 0 + S^3 0) \times S^3 0$.

The idea of term sequence is the following sequence

$$[0, S0, SS0, SSS0, (S^2 0 + S^3 0), \underbrace{(S^2 0 + S^3 0) \times S^3 0}_{\text{required term}}]$$

We can encode this sequence as : $\prod_{i=0}^{5} \pi_i^{l_i}$, where l_i is encoding of ith element in the sequence.

Encoding using p.r.

Genesis of a term:

Ch-19, 20

- $\text{Term}_{n-p.r.}(n) = \exists x\ (\text{TermsSq}(x) \wedge n = exf(x, \text{len}(x) - 1))$
- But can this be captured by p.r. ?
- What do you think?

- $\text{Term}(n) = \exists x \leq (\pi_l^n)^l\ (\text{TermsSq}(x) \wedge n = exf(x, \text{len}(x) - 1))$, where $l := \text{len}(n)$

How??

Encoding using p.r.

Sentence Genesis

Ch-19, 20

- $\mathsf{Atom}(n) := (n = \ulcorner \perp \urcorner) \vee \exists x, y < n(\mathsf{Term}(x) \wedge \mathsf{Term}(y) \wedge n = x \circ \ulcorner = \urcorner \circ y)$
- Can you define $\mathsf{Wff}(n)$? It's similar to how $\mathsf{Term}(n)$ was crafted
- $\mathsf{Sent}(n) := \mathsf{Wff}(n) \wedge \forall i \leq len(n)(\mathsf{Var}(\mathsf{exf}(n, i)) \to \mathsf{Bound}(\mathsf{exf}(n, i), i, n))$
    - Where,
    - $\mathsf{Bound}(v, i, n) := ?$

Encoding using p.r.

Sentence Genesis

Ch-19, 20

- $\text{Atom}(n) := (n = \ulcorner \bot \urcorner) \vee \exists x, y < n(\text{Term}(x) \wedge \text{Term}(y) \wedge n = x \circ \ulcorner = \urcorner \circ y)$

- Can you define $\text{Wff}(n)$? It's similar to how $\text{Term}(n)$ was crafted

- $\text{Sent}(n) := \text{Wff}(n) \wedge \forall i \leq len(n)(\text{Var}(\text{exf}(n, i)) \rightarrow \text{Bound}(\text{exf}(n, i), i, n))$

  - Where,

  - $\text{Bound}(v, i, n) := \text{Wff}(n) \wedge \text{Var}(v) \wedge$
    $\exists a, b, c < x\Big(x = a \circ \ulcorner \forall \urcorner \circ v \circ b \circ c \wedge \text{Wff}(b) \wedge$
    $\big(\text{len}(a) \leq i \wedge i \leq (\text{len}(a) + \text{len}(b) + 1)\big)\Big)$

Encoding using p.r.

Ch-19, 20

Sentence Genesis

- $\text{Atom}(n) := (n = \ulcorner \perp \urcorner) \vee \exists x, y < n(\text{Term}(x) \wedge \text{Term}(y) \wedge n = x \circ \ulcorner = \urcorner \circ y)$

- Can you define $\text{Wff}(n)$? It's similar to how $\text{Term}(n)$ was crafted

- $\text{Sent}(n) := \text{Wff}(n) \wedge \forall i \leq len(n)(\text{Var}(\text{exf}(n, i)) \rightarrow \text{Bound}(\text{exf}(n, i), i, n))$

  - Where,

  - $\text{Bound}(v, i, n) := \text{Wff}(n) \wedge \text{Var}(v) \wedge$
    $\exists a, b, c < x\Big(x = a \circ \ulcorner \forall \urcorner \circ v \circ b \circ c \wedge \text{Wff}(b) \wedge$
    $\big(\text{len}(a) \leq i \wedge i \leq (\text{len}(a) + \text{len}(a) + 1)\big)\Big)$