Sep 24, 2026

Gödel's Incompleteness Theorems

A formal study of introspection

Note: In the book Smith has assigned 0 to be true and 1 to be false. Keep that in mind while going through the book. Particularly Ch-14. The notion of characteristic function for a relation will be swapped.

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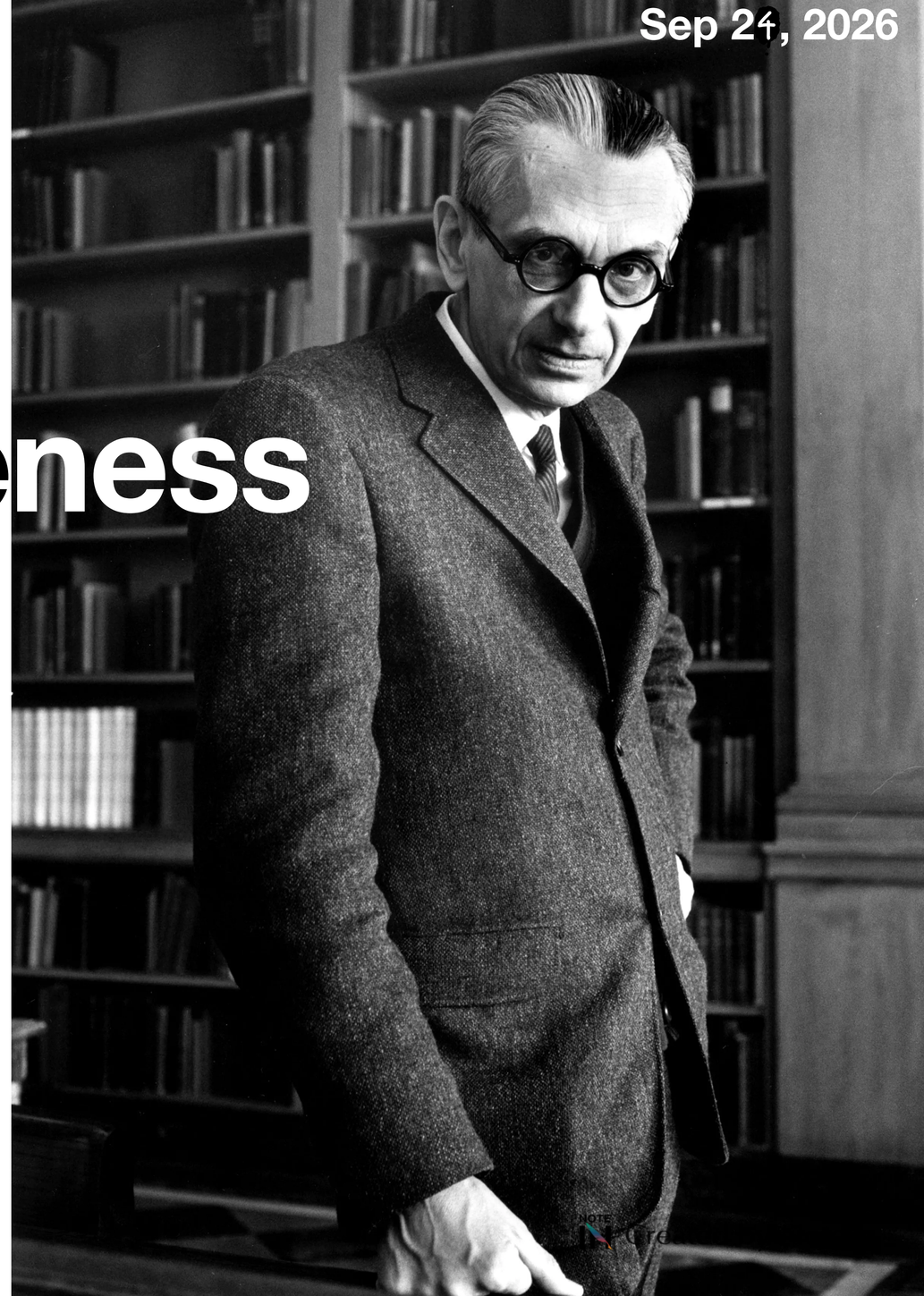

Pulling the meta-theory inside formal theory

Some tools…

Prime numbers:2, 3, 5, 7, 11, 13, 17 , 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139…

- Symbols we have:

$$\bot\,,\ \rightarrow,\ \forall,\ =,\ 0,\ S,\ +,\ \times,\ (,\ )$$

- We can put Gödel numbering as $\overline{1}$, $\overline{3}$, $\overline{5}$, $\overline{7}$, $\overline{9}$, $\overline{11}$, $\overline{13}$, $\overline{15}$, $\overline{17}$ and $\overline{19}$. Rest of the Odd numbers are for variables. Something like this: $\ulcorner v_i \urcorner := 2 \times i + 21$

- Then for a formula $\ulcorner \forall v_0(v_1 = S00 \times v_0 \rightarrow \bot\,) \rightarrow \bot \urcorner =$

$$2^{\ulcorner\forall\urcorner} \times 3^{\ulcorner v_0\urcorner} \times 5^{\ulcorner(\urcorner} \times 7^{\ulcorner v_1\urcorner} \times 11^{\ulcorner=\urcorner} \times 13^{\ulcorner S\urcorner} \times 17^{\ulcorner 0\urcorner} \times 19^{\ulcorner 0\urcorner} \times 23^{\ulcorner v_0\urcorner} \times 29^{\ulcorner\rightarrow\urcorner} \times 31^{\ulcorner\bot\urcorner} \times 37^{\ulcorner)\urcorner} \times 41^{\ulcorner\rightarrow\urcorner} \times 43^{\ulcorner\bot\urcorner}$$

$$= 2^5 \times 3^{21} \times 5^{17} \times 7^{23} \times 11^7 \times 13^{11} \times 17^9 \times 19^9 \times 23^{21} \times 29^3 \times 31^1 \times 37^{19} \times 41^3 \times 43^1$$

Revisiting to p.r. functions

For loops in disguise… (P.pp-108) (ch-14)

- A relation $P(\bar{x})$ is a p.r. relation, if the characteristic function is a p.r. function.
- If $f(\bar{x})$ is an n-place p.r. function, then the corresponding relation expressed by $f(\bar{x}) = y$ is an $n + 1$-place p.r. relation.
- Any truth-functional combination of p.r. relations is p.r.
- Any relation defined from a p.r. property or relation by bounded quantifications is also p.r.
- If P is a p.r. relation, then the function $f(n) = (\mu x \leq n)P(x)$ is p.r And generalizing, suppose that $g(n)$ is a p.r. function; then $f'(n) = (\mu x \leq g(n))P(x)$ is also p.r.
- Any function defined by cases from other p.r. functions is also p.r.

Revisiting to p.r. functions (Ch-14)

For loops in disguise… (P.pp-111)

- The relations $m = n$, $m < n$ and $m \leq n$ are primitive recursive. → First define monus in p.r. Then use it to define modulus. Also define signum function.
- The relation $m \mid n$ that holds when m is divides n is primitive recursive.
  ↳ $\exists y < m \,[\, n = m \times y \wedge 0 < m \wedge 0 < y \,]$
- Let $\mathsf{Prime}(n)$ be true just when n is a prime number. Then $\mathsf{P}(\,\cdot\,)$ is a p.r. property. Try it yourself
- List the primes as $\pi_0, \pi_1, \pi_2, \ldots$ Then the function $\pi(n)$ whose value is π_n is p.r.
  ↳ Use the fact that any prime greater than k is bounded by $k! + 1$
- Let $\mathsf{exf}(n, i)$ be the possibly zero exponent of π_i — the $i + 1$-th prime — in the factorization of n. Then $\mathsf{exf}(\,\cdot\,)$ is a p.r. function.
- Let $\mathsf{len}(0) = \mathsf{len}(1) = 0$; and when $n > 1$, let $\mathsf{len}(n)$ be the 'length' of n's factorization, i.e. the number of distinct prime factors of n. Then $\mathsf{len}(\,\cdot\,)$ is again a p.r. function.

Revisiting to p.r. functions

For loops in disguise...

- $exf(n,i) := (\mu x \leq n)\left(\pi_i^x \mid n \wedge \neg \pi_i^{x+1} \mid n\right)$

- $len(n) := \sum_{i=0}^{n} pd(i,n)$, where $pd(i,n) := Prime(i) \wedge i \mid n$

Try to write it out formally.

Encoding using p.r.

Our ‘glue’ is p.r.

Ch-19, 20

- There is a two-place concatenation function, standardly represented by $m \circ n$, such that (i) if m and n are the g.n. of two expressions then $m \circ n$ is the g.n. of the result of writing the first expression followed by the second, and (ii) this function is primitive recursive.

Example: Let $\ulcorner \forall v_1 \urcorner = m = 2^5 \times 3^{23}$ and
$\ulcorner (v_1 = 0 \rightarrow \bot\ ) \urcorner = n = 2^{17} \times 3^{23} \times 5^7 \times 7^9 \times 11^3 \times 13^1 \times 17^{19}$, then

$$\ulcorner \forall v_1 (v_1 = 0 \rightarrow \bot\ ) \urcorner = m \circ n = 2^5 \times 3^{23} \times 5^{17} \times 7^{23} \times 11^7 \times 13^9 \times 17^3 \times 19^1 \times 23^{19}$$

This concatenation function $\circ(\cdot, \cdot)$ can be written as: (∘ forms a monoid on ℕ, with 0 being identity)

$$m \circ n = (\mu x \leq \pi_{m+n}^{m+n})[(\forall i < \mathrm{len}(m))\{\mathrm{exf}(x, i) = \mathrm{exf}(m, i)\} \wedge$$
$$(\forall i < \mathrm{len}(n))\{\mathrm{exf}(x, i + \mathrm{len}(m)) = \mathrm{exf}(n, i)\}]$$

Why?

Encoding using p.r.

Ch-19, 20

Genesis of a term:

- The function num : $\{0\} \cup \{S^n 0 \mid n > 0\} \to \{\ulcorner\varphi\urcorner \mid \varphi \in \mathscr{L}_{PA}\}$ is p.r:
  $\mathsf{num}(0) = \ulcorner 0 \urcorner$
  $\mathsf{num}(Sx) = \ulcorner S \urcorner \circ \mathsf{num}(x)$

- The function $Diag : \mathscr{L}_{PA} \to \mathscr{L}_{PA}; \varphi \mapsto \exists v_1(v_1 = \ulcorner\varphi\urcorner \wedge \varphi)$ can be turned to a function on Gödel encodings: (There are huge elephants in the room lurking here.Can you see?)
  $\mathsf{diag}(n) = \ulcorner \exists v_1(v_1 = \urcorner \circ \mathsf{num}(0) \circ \ulcorner \wedge \urcorner \circ n \circ \ulcorner)\urcorner$

- $\mathsf{Var}(n) = \exists x \leq n \ (n = 2 \times x + 17)$: For checking if n is a encoding of a variable

- $\mathsf{TermSeq}(n) := \forall k < \mathsf{len}(n)\{\mathsf{exf}(n,k) = \ulcorner 0 \urcorner \vee \mathsf{Var}(exf(n,k)) \vee$
  $\exists j < k \ (\mathsf{exf}(n,k) = \ulcorner \mathsf{S} \urcorner) \circ \mathsf{exf}(n,j)) \vee$
  $\exists i,j < k \ (\mathsf{exf}(n,k) = \ulcorner(\urcorner \circ \mathsf{exf}(n,i) \circ \ulcorner + \urcorner \circ \mathsf{exf}(n,j) \circ \ulcorner)\urcorner) \vee$
  $\exists i,j < k \ (\mathsf{exf}(n,k) = \ulcorner(\urcorner \circ \mathsf{exf}(n,i) \circ \ulcorner \times \urcorner \circ \mathsf{exf}(n,j) \circ \ulcorner)\urcorner)\}$

Encoding using p.r.

Genesis of a term:

Ch-19, 20

- $\text{Term}_{n-p.r.}(n) = \exists x\ (\text{TermsSq}(x) \wedge n = exf(x, \text{len}(x) - 1))$
- But can this be captured by p.r. ?
- What do you think?

↳ TermSequence : Say we want to encode $(S^2 0 + S^3 0) \times S^3 0$.

The idea of term sequence is the following sequence

$[\,0,\ S0,\ SS0,\ SSS0,\ (S^2 0 + S^3 0),\ \underbrace{(S^2 0 + S^3 0) \times S^3 0}_{\text{required term}}\,]$

We can encode this sequence as : $\prod_{i=0}^{5} \pi_i^{l_i}$, where l_i is encoding of i^{th} element in the sequence.

Encoding using p.r.

Genesis of a term:

Ch-19, 20

- $\text{Term}_{n-p.r.}(n) = \exists x \ (\text{TermsSq}(x) \wedge n = exf(x, \text{len}(x) - 1))$
- But can this be captured by p.r. ?
- What do you think?

- $\text{Term}(n) = \exists x \leq (\pi_l^n)^l \ (\text{TermsSq}(x) \wedge n = exf(x, \text{len}(x) - 1))$, where $l := \text{len}(n)$

How??

Encoding using p.r.

Sentence Genesis

Ch-19, 20

- $\mathsf{Atom}(n) := (n = \ulcorner \bot \urcorner) \vee \exists x, y < n(\mathsf{Term}(x) \wedge \mathsf{Term}(y) \wedge n = x \circ \ulcorner = \urcorner \circ y)$
- Can you define $\mathsf{Wff}(n)$? It's similar to how $\mathsf{Term}(n)$ was crafted
- $\mathsf{Sent}(n) := \mathsf{Wff}(n) \wedge \forall i \leq len(n)(\mathsf{Var}(\mathsf{exf}(n, i)) \rightarrow \mathsf{Bound}(\mathsf{exf}(n, i), i, n))$
    - Where,
    - $\mathsf{Bound}(v, i, n) := ?$

Encoding using p.r.

Sentence Genesis

Ch-19, 20

- $\text{Atom}(n) := (n = \ulcorner \bot \urcorner) \vee \exists x, y < n(\text{Term}(x) \wedge \text{Term}(y) \wedge n = x \circ \ulcorner = \urcorner \circ y)$
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    - Where,
    - $\text{Bound}(v, i, n) := \text{Wff}(n) \wedge \text{Var}(v) \wedge$
      $\exists a, b, c < x \Big(x = a \circ \ulcorner \forall \urcorner \circ v \circ b \circ c \wedge \text{Wff}(b) \wedge$
      $\big(\text{len}(a) \leq i \wedge i \leq (\text{len}(a) + \text{len}(b) + 1)\big) \Big)$

Encoding using p.r.

Ch-19, 20

Sentence Genesis

- $\text{Atom}(n) := (n = \ulcorner \bot \urcorner) \vee \exists x, y < n(\text{Term}(x) \wedge \text{Term}(y) \wedge n = x \circ \ulcorner = \urcorner \circ y)$
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    - Where,
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      $\exists a, b, c < x \Big(x = a \circ \ulcorner \forall \urcorner \circ v \circ b \circ c \wedge \text{Wff}(b) \wedge$
      $\big(\text{len}(a) \leq i \wedge i \leq (\text{len}(a) + \text{len}(a) + 1)\big) \Big)$

Encoding using p.r.

Inference Rules

- $\mathrm{MP}(m,n,o) := \{m = \ulcorner(\urcorner \circ n \circ \to \circ o \circ \ulcorner)\urcorner\} \wedge \mathrm{Wff}(n) \wedge \mathrm{Wff}(o)$
- $\mathrm{Univ}(m,n) := (\exists x \leq n)(\mathrm{Var}(x) \wedge \mathrm{Wff}(m) \wedge \{n = \ulcorner\forall\urcorner \circ x \circ m\})\,.$
- $\mathrm{Proofseq}(n) := (\forall k < \mathrm{len}(n))\{\mathrm{Axiom}(\mathrm{exf}(n,k)) \vee$

  $(\exists i,j \leq k)\mathrm{MP}(\mathrm{exf}(n,i),\mathrm{exf}(n,j),\mathrm{exf}(n,k)) \vee$

  $(\exists j \leq k)\mathrm{Univ}(\mathrm{exf}(n,j),\mathrm{exf}(n,k))\}$
- $\mathrm{Prf}(m,n) := \mathrm{Proofseq}(m) \wedge n = \mathrm{exf}(m,\mathrm{len}(m)-1) \wedge \mathrm{Sent}(n)$ Δ_0

$[\mathrm{Prov}(n) = \exists y\ \mathrm{Prf}(y,n) - \Sigma_1$

→ Σ_1 completeness of PA

→ P.R. functions / relations can be captured in PA using Σ_1-sentences.

For any $n \in \mathbb{N}$, $PA \vdash \forall x (x \leq \bar{n} \rightarrow x = 0 \vee x = \bar{1} \vee \cdots \vee x = \bar{n})$

Proof: Induction on $n \in \mathbb{N}$.

Base Case: $n = 0$, $PA \vdash \forall x (x \leq \bar{0} \rightarrow x = 0)$

$PA \vdash a \leq \bar{0}$ $\quad PA \vdash \forall x (x = 0 \vee \exists y (Sy = x))$

$PA \vdash \exists v (v + a = 0)$ $\quad a = 0 \vee \exists y\, Sy = a$.

$\vdash b + a = 0$ $\quad \vdash b + Sc = 0 \Rightarrow \vdash S(b + c) = 0$

Inductive case: $\rightarrow PA \vdash \forall x \left(x \leq \bar{n} \rightarrow \bigvee_{i=0}^{n} \{x = i\} \right)$

$PA \vdash a \leq \overline{n+1}$

$PA \vdash \exists v \, (v + a = \overline{n+1})$

$PA \vdash b + a = \overline{n+1}$

$- a = Sc$, $PA \vdash b + Sc = S(b+c) = \overline{n+1}$

$= \cdot S\,\bar{n}$

$PA \vdash b + c = \bar{n}$

$PA \vdash c \leq \bar{n} \xRightarrow{I.H} PA \vdash \bigvee_{i=1}^{\bar{n}} \{c = i\}$

$PA \vdash a = 0 \vee \bigvee_{i=1}^{\overline{n+1}} \{a = i\}$

$PA \vdash \bigvee_{i=1}^{\overline{n+1}} \{a = i\}$

$PA \vdash a \leq \overline{n+1} \rightarrow \bigvee_{i=0}^{\overline{n+1}} \{a = i\}$

Note this will work for Q will also work.

$\overline{n} \leq \overline{m} := \exists u(\overline{n} + u = \overline{m})$

PA is order-adequate

(Ch-11)

"By the power of Grayskull, I have the power of $\leq$"

1. $PA \vdash \forall x(0 \leq x)$
2. For any n, $PA \vdash \forall x(\{x = 0 \vee x = \overline{1} \vee \ldots \vee x = \overline{n}\} \leftrightarrow x \leq \overline{n})$
3. For any n, if $PA \vdash \varphi(0)$, $PA \vdash \varphi(\overline{1})$ … $PA \vdash \varphi(\overline{n})$, then $PA \vdash (\forall x \leq \overline{n})\varphi(x)$
4. For any n, if $PA \vdash \varphi(0)$, or $PA \vdash \varphi(\overline{1})$, or … $PA \vdash \varphi(\overline{n})$ then $PA \vdash (\exists x \leq \overline{n})\varphi(x)$
5. For any n, $PA \vdash \forall x(x \leq \overline{n} \rightarrow x \leq S\overline{n})$
6. For any n, $PA \vdash \forall x(\overline{n} \leq x \rightarrow (\overline{n} = x \vee S\overline{n} \leq x))$
7. For any n, $PA \vdash \forall x(x \leq \overline{n} \vee \overline{n} \leq x)$
8. For any $n > 0$, $PA \vdash (\forall x \leq \overline{n-1})\varphi(x) \rightarrow (\forall x \leq \overline{n})(x \neq \overline{n} \rightarrow \varphi(x))$

$\vdash \sigma = t + t'$, $\vdash \bar{i} = \bar{a} + b$ need to show.

Lemma If $i + j = k$, then $\vdash \bar{i} + \bar{j} = \bar{k}$

$4 + 6 = 10 \quad \vdash \bar{4} + \bar{6} = \overline{10}$

Proof: Induction on j.

σ	τ
m	n

$\vdash \bar{i} = \bar{j}$ or $i = j$ PA $\vdash \underline{\sigma} = \underline{\tau}$.

Suppose $N \models \neg(\sigma = \tau)$ $\vdash \sigma = \tau$

$n < m$, $k = (n - m) - 1$

$\underline{1} \cdot 0 \neq S\bar{k}$

$PA \vdash \neg\exists v\,(\sigma + v = \tau)$

$PA \vdash \sigma \leq \tau$

$N \models \neg(\sigma \leq \tau) \Longrightarrow PA \vdash \neg(\sigma \leq \tau)$

$PA \vdash \neg\exists v\,(\sigma + v = \tau)$

start

$PA \vdash \exists v\,(\sigma + v = \tau) \rightsquigarrow \bot$

$N \models \sigma \leq \tau \Rightarrow PA \vdash \sigma \leq \tau$

$N \models \neg(\sigma \leq \tau) \Rightarrow PA \vdash \neg(\sigma \leq \tau)$

PA decides Δ_0

The key is bounded quantification

For the inductive cases we need to use the power of the Order-adequacy from earlier. (P.pp-94)

$\neg\varphi := (\forall x \leq \bar{n})\,\psi(x)$

Case 1: $\mathbb{N} \models (\forall x \leq \bar{n})\,\psi(x)$

$\mathbb{N} \models \psi(0)$, $\mathbb{N} \models \psi(\bar{1})$ ....

$\mathbb{N} \models \psi(\bar{n})$

By I.H

$PA \vdash \psi(\bar{i})$, where $i \leq n$

So from Order-adequacy result 3, $PA \vdash (\forall x \leq \bar{n})\,\psi(x)$

Case 2: $\mathbb{N} \models \neg((\forall x \leq \bar{n})\,\psi(x))$ there is a $m \in \mathbb{N}$ s.t

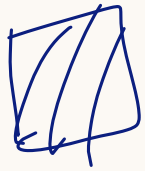
$m < n$ and $\mathbb{N} \models \neg\psi(\bar{m})$ $\xrightarrow{\text{I.H}}$ $PA \vdash \neg\psi(\bar{m})$

From Order-adequacy result 4, $PA \vdash (\exists x \leq \bar{n})\, \neg\varphi(x)$

$PA \vdash \neg((\forall x \leq \bar{n})\, \varphi(x))$

- $\varphi := (\exists x \leq \bar{n})\, \varphi(x)$

Similar reasoning as above.



→ Can be switched to PA itself.

PA is Σ_1 – Complete

(Ch-12)

$\varphi \in \Sigma_1$, I can show $PA \vdash \varphi$, whenever, $\mathbb{N} \models \varphi$.

↳ Completeness

PA proves all true formula of Σ_1 type

Proof style : Structural induction. Very similar to previous one. (P.pp-93)

Base case: Δ_0 sentences.

Inductive cases:

- Boolean cases are straight forward.
- $\mathbb{N} \models (\forall x \leq \bar{n})\, \varphi(x)$ $\quad \mathbb{N} \models \varphi(\bar{i})$, for all $i \leq n$.

  By I.H $PA \vdash \varphi(\bar{i})$, for all $i \leq n$.

  $PA \vdash (\forall x \leq \bar{n})\, \varphi(x)$ [Order adequacy of PA]
- Similarly for $(\exists x \leq \bar{n})\, \varphi(x)$.

– $\mathbb{N} \models \exists x\, \varphi(x)$, $\mathbb{N} \models \varphi(\bar{m})$ for some $m \in \mathbb{N}$.
By I.H, $PA \vdash \varphi(\bar{m})$. Hence $PA \vdash \exists x\, \varphi(x)$.

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1. $Z(x,y) := (x = x \wedge y = 0)$ [zero fun]
2. $P^n_k(\bar{x},y) := \bigwedge_{i=1}^{n} (x_i = x_i) \wedge y = x_k$ [Proj. fun]
3. $Succ(x,y) := Sx = y$ [Succ. fun]
4. $h : \mathbb{N}^n \to \mathbb{N}$ $\quad g_k : \mathbb{N}^m \to \mathbb{N}$ $\quad 1 \le k \le n$

$H(\bar{x}^n, y)$ $\quad g_k(\bar{x}^m, y)$

Want to capture : $h(g_1(\bar{x}), g_2(\bar{x}) \cdots g_n(\bar{x}))$

$$\exists \bar{v}^n \left(H(\bar{v}, y) \bigwedge_{i=1}^{n} g_i(\bar{x}^m, v_i) \right) = H \circ g(\bar{x}^m, y)$$

5. for the recursive step we need β-function. We will represent the recursion as a list and use β-function to encode it. An example

$$f(0,m) = g(m)$$
$$f(Sx,m) = h(x, f(x,m), m)$$

$\longrightarrow$ $k_0, k_1, \ldots k_x$ such that $k_0 = g(m)$,
$k_{Su} = h(u, k_u, m)$, $\forall u < x$ and
$k_x = y$. So $y = f(x,m)$ where
f is defined as in the left hand side.

$\beta-$Function

A way to encode finite list

- There exists a n function $\beta(\cdot,\cdot,\cdot)$, given any finite sequence $k_0, k_1, k_2, \cdots, k_n$, there are c and d such that for $i \leq n$, $\beta(c,d,i) = k_i$.

- In $\mathscr{L}$, we can define $\beta(\cdot,\cdot,\cdot)$ as

$$\beta(c,d,i,y) = rm(c, S(d \times Si), y)$$

$y = c \bmod [d(i+1)+1]$

Hence you can use $\beta(\cdot,\cdot,\cdot)$ function to encode a finite list