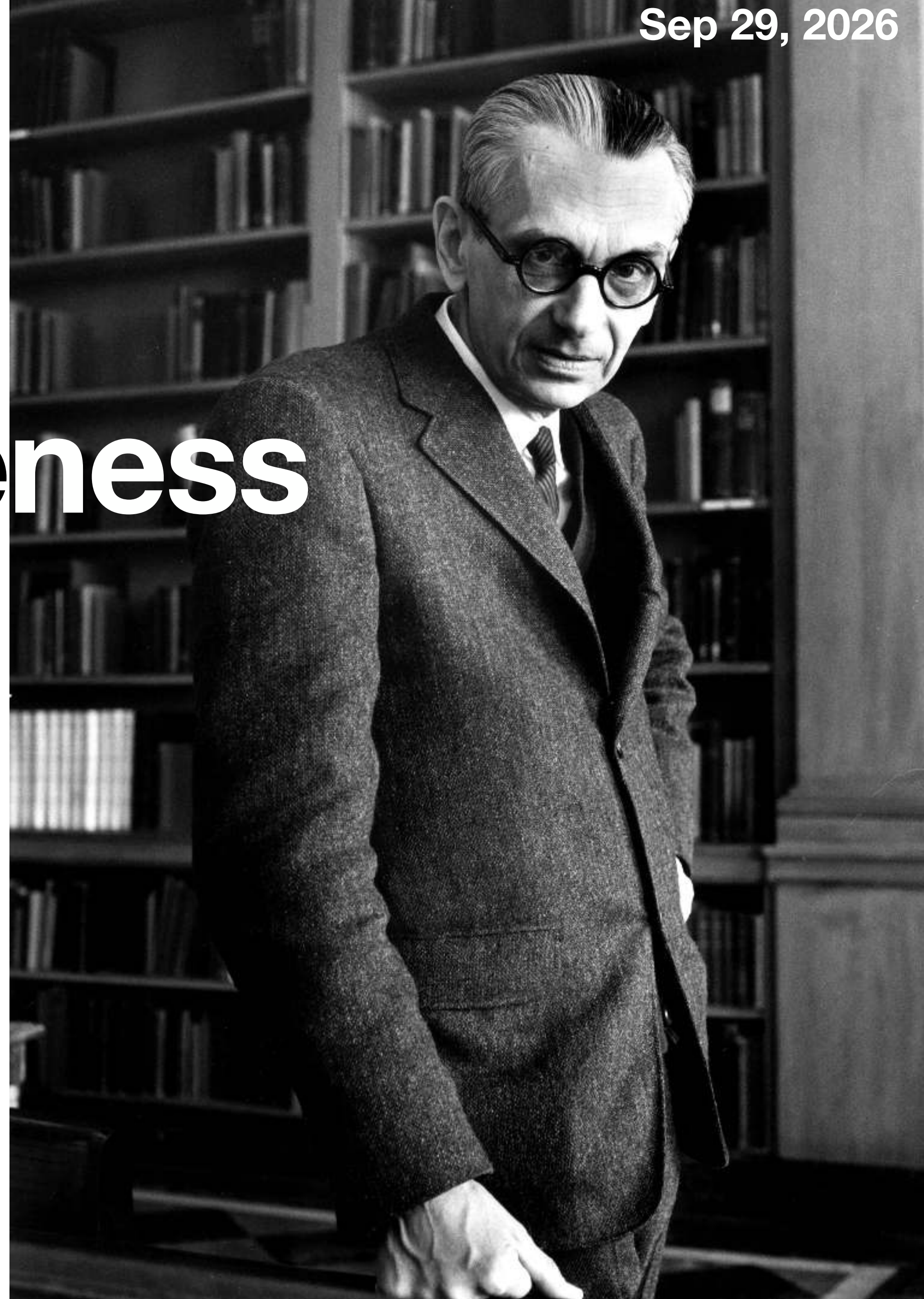


Gödel's Incompleteness Theorems

A formal study of introspection

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Peano Arithmetic

‘God made the natural numbers, all else is the work of man’ —Kronecker

1. $\forall x(0 \neq Sx)$

2. $\forall x \forall y(Sx = Sy \rightarrow x = y)$

→ 3. $\forall x(x + 0 = x)$

4. $\forall x \forall y(x + Sy = S(x + y))$

5. $\forall x(x \times 0 = 0)$

6. $x \times Sy = (x \times y) + x$

7. Induction Schema: $\varphi(0) \rightarrow \forall x(\varphi(x) \rightarrow \varphi(Sx)) \rightarrow \forall x\varphi$

$\mathbb{Q} \not\models \forall x (0 + x = x)$

Robinson Arithmetic

$(\mathbb{Q}: \forall x(\neg(x = 0) \rightarrow \exists y(x = Sy)))$

Primitive Recursive(p.r) Functions

A collection of functions of form $f : \mathbb{N}^n \rightarrow \mathbb{N}$ for some $n \in \mathbb{N}$

- Every $Z : \mathbb{N}^n \rightarrow \mathbb{N}$ which are identically zero is in p.r.
- For each $1 \leq k \leq n$, the function $P_k^n : \mathbb{N}^n \rightarrow \mathbb{N}$ defined by $(x_1, x_2, x_3, \dots, x_n) \mapsto x_k$ is in p.r.
- $S : \mathbb{N} \rightarrow \mathbb{N}$ defined as $n \mapsto n + 1$ is in p.r.
- If $f : \mathbb{N}^n \rightarrow \mathbb{N}$ is in p.r and for each $1 \leq k \leq n$, $g_k : \mathbb{N}^m \rightarrow \mathbb{N}$ is in p.r, then $h : \mathbb{N}^m \rightarrow \mathbb{N}$ is in p.r where h is defined by

$$h(x_1, x_2, \dots, x_m) = f(g_1(x_1, x_2, \dots, x_m), g_2(x_1, x_2, \dots, x_m), \dots, g_n(x_1, x_2, \dots, x_m))$$

$$\text{Notation: } h(x_1, x_2, \dots, x_m) = f \circ (g_1, g_2, g_3, \dots, g_n)(x_1, x_2, \dots, x_m)$$

- If $g : \mathbb{N}^{n+1} \rightarrow \mathbb{N}$ and $h : \mathbb{N}^{n-1} \rightarrow \mathbb{N}$ are both in p.r, then $f : \mathbb{N}^n \rightarrow \mathbb{N}$ is in p.r where,

Primitive Recursive(p.r) Functions

$$\bullet f(x+1, x_1, \dots, x_{n-1}) = \begin{cases} h(x_1, x_2, \dots, x_{n-1}) & \text{If } x = 0 \\ g(x, f(x, x_1, x_2, \dots, x), x_1, x_2, \dots, x_{n-1}) & \text{If } x \geq 1 \end{cases}$$

First Order Logic *Ch-5,16*

Function *capturing* in a theory T and function *expressing* in a Language

- Let, $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function from natural numbers to natural numbers.
- We say a theory T captures $f : \mathbb{N} \rightarrow \mathbb{N}$ by a formula $\varphi(\cdot, \cdot)$ if:

1. $T \vdash \varphi(\bar{m}, \bar{n})$, if $f(m) = n$.

2. $T \vdash \exists! y \varphi(\bar{m}, y)$

Fact 1: PA can prove true Σ_1 sentences.

Fact 2: p.r functions are expressed by Σ_1

Fact 3: Hence PA can capture p.r functions by Σ_1 sentences

- We say function is expressible in $\mathcal{L}$ if:

A. $\mathbb{N} \models \varphi(\bar{m}, \bar{n})$, whenever $f(m) = n$

B. $\mathbb{N} \models \neg \varphi(\bar{m}, \bar{n})$, whenever $f(m) \neq n$

(Ch-15)

β -Function

$$f(0, m) = g(m)$$
$$f(Sx, m) = h(x, f(x, m), m)$$

A way to encode finite list

Using finite list formalism this would look like: $k_0 = g(m); \forall u < \omega \ k_{Su} = h(u, k_u, m);$
 $k_\omega = y$ where $f(x, m) = y$.

• There exists a function $\beta(\cdot, \cdot, \cdot)$, given any finite sequence $k_0, k_1, k_2, \dots, k_n$, there are c and d such that for $i \leq n$, $\beta(c, d, i) = k_i$.

• In $\mathcal{L}$, we can define $\beta(\cdot, \cdot, \cdot)$ as

$$\beta(c, d, i, y) = rm(c, S(d \times Si), y)$$

Hence you can use $\beta(\cdot, \cdot, \cdot)$ function to encode a finite list

$$f(0, m) = 0$$

$$f(2, 3)$$

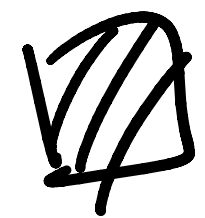
$$\Rightarrow f(Sn, m) = f(n, m) + m$$
$$\left[f(0, 3), \right.$$
$$\left[0, f(0, 3) + 3 \right.$$
$$\rightarrow [0, 3, \dots, 6]$$

β -Function (ch-15)

A way to encode finite list

- Lemma 1: Suppose x, y are relatively prime. Then there is a number $b < y$ such that $\text{rm}(bx, y) = 1$.

- Proof: $\text{gcd}(x, y) = 1$ $bx = 1 \pmod y$
 $b_1 < b_2 < y$ $b_1x = b_2x \pmod y$
 $y \mid (b_1 - b_2)x$ but $\rightarrow \leftarrow$
 $y \nmid x$ and $y \nmid b_1 - b_2$



β -Function (ch-15)

A way to encode finite list

- Lemma 2(CR Theorem): Suppose $m_0, m_1, m_2, \dots, m_n$ are pairwise relatively prime. Then for any sequence of numbers $k_0, k_1, k_2, \dots, k_n$ (where for each $i \leq n$, $k_i \leq m_i$) there is a number c such that $k_i = \text{rm}(c, m_i)$.

$$c = k_i \bmod m_i \quad M_i = \prod_{j \neq i} m_j \quad \text{gcd}(M_i, m_i) = 1$$

- Proof:

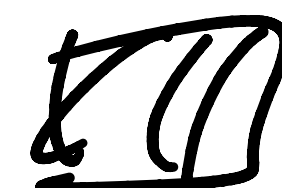
$$\forall i \exists b_i \quad b_i M_i = 1 \bmod m_i$$

$$c = \sum_{j=0}^n k_j b_j M_j$$

$$m_i \mid k_j b_j M_j \text{ when } j \neq i$$

$$\text{at } j=i \quad k_i = k_i b_i M_i \bmod m_i$$

$$c = k_i \bmod m_i$$



β -Function (ch-15)

A way to encode finite list

- Gödel's β -function Lemma: For any sequence of numbers $k_0, k_1, k_2, \dots, k_n$, there are numbers c, d such that $\text{rm}(c, d(i+1) + 1) = k_i$.

- Proof: $u = \max(n+1, k_0, \dots, k_n)$ $\underline{d} = u!$. For $i \leq n$

$$m_i = d(i+1) + 1 \quad \begin{cases} m_x = da + 1 \\ m_y = db + 1 \end{cases} \quad \begin{array}{l} \text{some prime } p \\ p \mid m_x \text{ \& } p \nmid m_y \\ 1 \leq a < b \leq n+1 \end{array}$$

$$p \mid d(b-a) \text{ but } (b-a) \nmid d$$

so this means $p \nmid d$

m_i are pairwise relatively prime.

by C.R. there is a c such that $c = k_i \bmod (d \times (i+1))$
 $+ 0 \leq i \leq n. \quad \square$

$$\beta(c, d, i) = k_i$$

β -Function

(ch-15)

A way to encode finite list

A particular example to study:

$$f(0, m) = g(m)$$

$$f(Sx, m) = h(x, f(x, m), m)$$

(go with the simplest case)

- $B(c, d, i, y)$ expresses $\beta(c, d, i) = y$.

$$(1) \beta(c, d, 0) = k_0 \quad (2) \forall u < x \quad \beta(c, d, Su) = h(u, \beta(c, d, u), m)$$

$$(3) \beta(c, d, x) = y \text{ where } f(x, m) = y.$$

Inside L_{PA} one can write like this.

$$\exists c \exists d \left[\exists a B(c, d, 0, a) \wedge G(m, a) \wedge \forall u < x \left(u \neq 0 \vee \exists s \exists t (B(c, d, u, s) \wedge B(c, d, Su, t) \wedge H(u, s, m, t)) \right) \right] \wedge B(c, d, x, y) \equiv F(x, m, y)$$

β —Function in action (ch-15)

Σ_1 expresses

- Zero Function : $Z(x, y) := (x = x \wedge y = 0)$
- Successor Function : $S(x, y) := (y = S(x))$
- Projections : $Proj_k^n(x_1, x_2, x_3, \dots, x_n, y) := \left(y = x_k \wedge \bigwedge_{i=1, i \neq k}^n (x_i = x_i) \right)$
- Compositions: $H \cdot G(x_1, \dots, x_m, y) := \exists y_1 \dots \exists y_n \left(H(y_1, \dots, y_n, y) \wedge \bigwedge_i G_i(x_1, \dots, x_m, y_i) \right)$
- Primitive recursive step: $\exists c \exists d \left[\exists a B(c, d, 0, a) \wedge G(m, a) \wedge \forall u \leq n \left(u = n \vee \exists s \exists t (B(c, d, u, s) \wedge B(c, d, Su, t) \wedge H(u, s, m, t)) \wedge B(c, d, u, y) \right) \right]$

β -Function in action

Note: for a given $\varphi(x, y)$,

$$\tilde{\varphi}(x, y) := \varphi(x, y) \wedge \forall z \leq y (\varphi(x, z) \rightarrow z = y)$$

Σ_1 captures

- Zero Function : $Z(x, y) := (x = x \wedge y = 0)$

- Successor Function : $S(x, y) := (y = S(x))$

- Projections : $Proj_k^n(x_1, x_2, x_3, \dots, x_n, y) := \left(y = x_k \wedge \bigwedge_{i=1, i \neq k}^n (x_i = x_i) \right)$

- Compositions: $H \cdot G(x_1, \dots, x_m, y) := \exists y_1 \dots \exists y_n \left(H(y_1, \dots, y_n, y) \wedge \bigwedge_i G_i(x_1, \dots, x_m, y_i) \right)$

- Primitive recursive step: Almost same as above, but

instead of $B(c, d, i, y)$ use $\tilde{B}(c, d, i, y)$

(Ch-17)

β -Function in action

Σ_1 captures

$$\tilde{B}(c, d, i, y) := B(c, d, i, y) \wedge \forall z \leq y \\ \rightarrow \underbrace{(B(c, d, i, z) \rightarrow z = y)}$$

(2) for every $m \in \mathbb{N}$, $PA \vdash \exists! y F(m, y) \rightarrow$ Try to prove this using induction on m .

Then we will have

“PA captures p.r. functions”

Let me know if there is some difficulty

$\Rightarrow PA$ is p.r. adequate.

Q is p.r. adequate

Q is Σ_1 -complete

$T \not\models \varphi$ and $T \not\models \neg \varphi$

$$T \cup \{\varphi\} = T^*$$

$$Q \not\models \forall x (0 + x = x)$$

$$Q \not\models \neg \forall x (0 + x = x)$$

$$\begin{aligned} & N^* \models Q \\ & N^* \models \neg \forall x (0 + x = x) \\ & \rightarrow \text{How do we create} \\ & \text{the counter model?} \end{aligned}$$

$$Q^* \not\models \forall x, y (Sx + y = S(x, y))$$

$$Q \vdash 0 + \bar{m} = \bar{m}, \text{ for every } m \in \mathbb{N}$$

$$f(0) = g \text{ and } f(su) = \underline{h}(f(u)) \quad f(\bar{m}) = \bar{n}$$

$$(1) \beta(c, d, 0) = g \quad (2) \text{ If } u < \bar{m} \quad \beta(c, d, su) = h(\beta(c, d, u))$$

$$(3) \beta(c, d, \bar{m}) = \bar{n}$$

$$\tilde{\beta}(c, d, 0, \bar{g}) \wedge \forall u \leq \bar{m} \left[u = \bar{m} \vee \exists v \exists w \right. \\ \left. \tilde{\beta}(c, d, u, v) \wedge \tilde{\beta}(c, d, su, w) \right. \\ \left. \wedge H(v, w) \right] \wedge$$

$$\tilde{\beta}(c, d, \bar{m}, \bar{n})$$

$$F(\bar{m}, \bar{n})$$

$$PA \vdash F(\bar{m}, \bar{n})$$

$$PA \vdash \exists! y F(\bar{m}, y) \iff PA \vdash \exists y F(\bar{m}, y) \wedge \forall e (F(\bar{m}, e) \rightarrow e = y)$$

$$PA \vdash \forall e (\underline{F(\bar{m}, e)} \rightarrow \underline{e = \bar{n}})$$

$$\underline{\underline{m=0}}$$

$$\underline{\tilde{B}(c, d, 0, e) \wedge B(c, d, 0, \bar{g})}$$

$$e = \bar{g}$$

$$n = g$$

$$\vdash F(\bar{n}, \textcircled{e})$$

$$\vdash e = \bar{g} \vdash e = n$$

$$\therefore PA \vdash F(\bar{n}, e) \rightarrow e = \bar{n}$$

$$\underline{\underline{m > 0}}$$

$$\exists u \exists v \tilde{B}(c, d, 0, u) \wedge \tilde{B}(c, d, 1, v)$$

$$\wedge H(v, w)$$

$$\varphi \rightarrow \tilde{\varphi}(x, y) := \varphi(x, y) \wedge \forall z \leq y$$

$$(\varphi(x, z) \rightarrow z = y)$$

$$PA \vdash \forall x \forall y (\tilde{\varphi}(x, \bar{n}) \wedge \tilde{\varphi}(x, y) \rightarrow \underline{y = \bar{n}})$$

Proof! $\tilde{\varphi}(a, \bar{n}) \wedge \underline{\tilde{\varphi}(a, b)}$

$$PA \vdash b \leq \bar{n} \vee \bar{n} \leq b \quad (\text{order adequacy})$$

Case 1: $\underline{b \leq \bar{n}} \quad \tilde{\varphi}(a, \bar{n})$

$$\forall z \leq \bar{n} (\varphi(a, z) \rightarrow z = \bar{n})$$

$$\vdash \varphi(a, b) \rightarrow b = \bar{n}$$

$$\vdash b = \bar{n}$$

case 2: $\bar{n} \leq b \implies b = \bar{n}$

$$\varphi(a, \bar{n}) \wedge \varphi(a, b) \rightarrow b = \bar{n}$$

$$\exists u \exists v \underbrace{\tilde{B}(c, d, 0, u) \wedge \tilde{B}(c, d, \underline{1}, \underline{u})}_{\bar{k}_1}$$

$$\wedge H(u, u)$$

$$\tilde{B}(c, d, 0, u) \wedge \tilde{B}(c, d, 0, \bar{g})$$

$$u = \bar{g}$$

$$\text{so } H(\bar{g}, \underline{u}) \quad H(\bar{g}, \bar{k}_1)$$

$$\tilde{B}(c, d, 1, \bar{k}_1)$$

$$\tilde{B}(c, d, 0, \underline{\bar{g}})$$

$$\begin{matrix} 0 \\ \downarrow \\ 1 \end{matrix}$$



Przf (m, n)