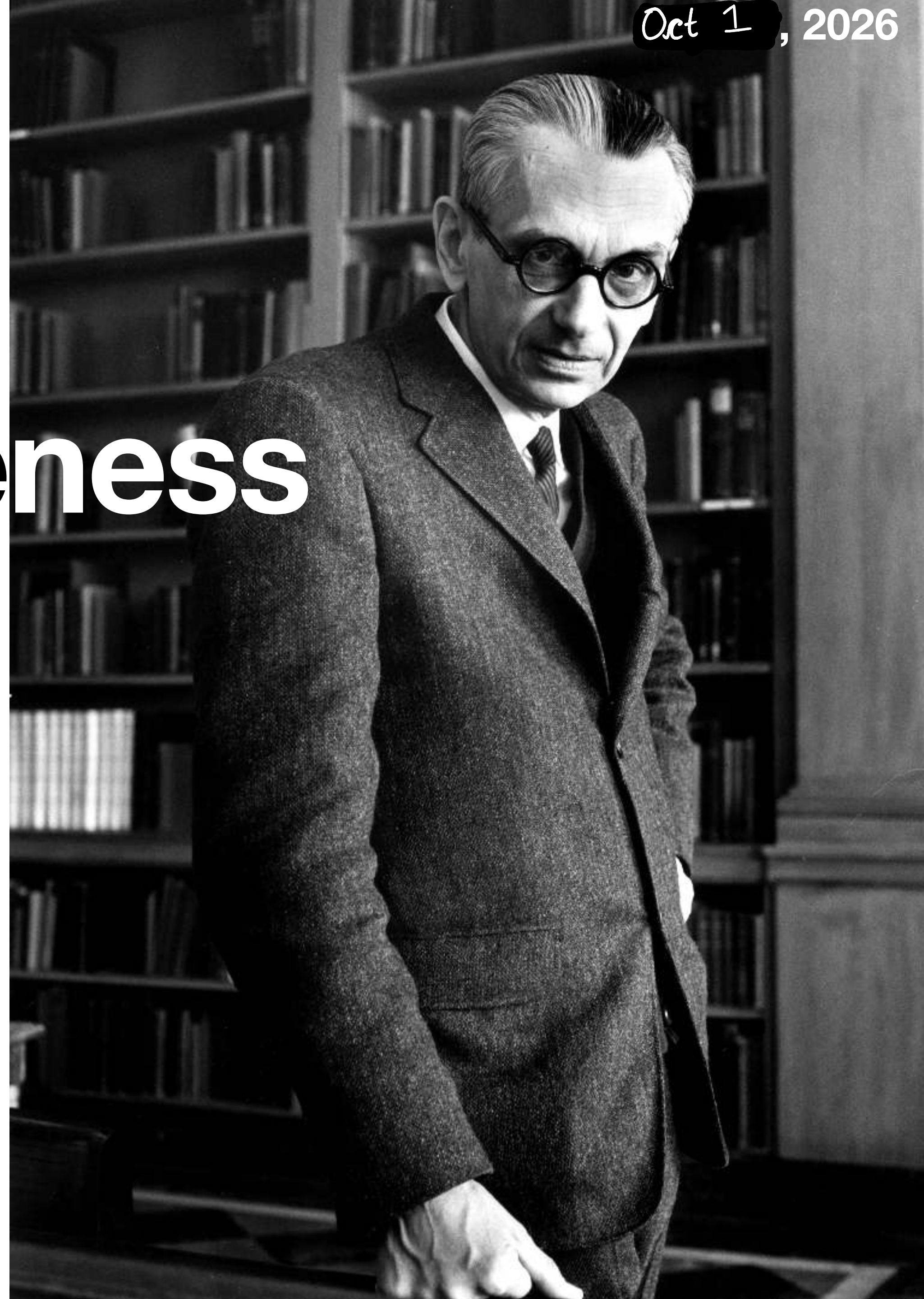


# Gödel's Incompleteness Theorems

A formal study of introspection

P.S: All the books can be  
found in the resources  
channel of discord server.

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$\varphi(\ulcorner \varphi \urcorner) \rightarrow$  diagonalized

Tell me what I did wrong here?

# Revisiting the $\text{Prf}(m, n)$ predicate (ch 21)

PA captures this through p.r function using  $\Sigma_1$  formula

- So we have  $\text{diag}(n)$ , which takes in g.n of a formula  $\varphi$  and gives us the g.n of the diagonalized version of  $\varphi$ ; where diagonalization of  $\varphi$  is  $\exists y (y = \ulcorner \varphi \urcorner \wedge \varphi)$ .
- We have the provability predicate as  $\text{Prov}(x) := \exists y \text{Prf}(y, x)$ , and we have  $U(y) := \neg \text{Prov}(\text{diag}(y))^*$  then we have the Gödel sentence as  $\mathcal{G} := \exists y (y = \ulcorner U \urcorner \wedge U(y))$   
 $\rightarrow \text{diag}(\cdot) \notin \mathcal{L}_{\text{PA}} \rightsquigarrow \text{Diag}(m, n) \rightsquigarrow \Sigma_1$
- In other words,  $\mathcal{G}$  is equivalent to  $\neg \exists x \text{Prf}(x, \text{diag}(\ulcorner U \urcorner))$ , but  $\ulcorner \mathcal{G} \urcorner = \text{diag}(\ulcorner U \urcorner)$ .
- Now we already know that:

If PA is sound then  $\text{PA} \not\models \mathcal{G}$  and  $\text{PA} \not\models \neg \mathcal{G}$ .

$\text{Diag} : \mathcal{L}_{\text{PA}} \rightarrow \mathcal{L}_{\text{PA}}$   
 $\varphi \mapsto \exists y (y = \ulcorner \varphi \urcorner \wedge \varphi)$

# Syntactic Argument

$$G_2 \sim \neg \exists x \text{Prf}(x, \text{diag}(\ulcorner \bar{v} \urcorner))$$

If PA is consistent then  $PA \not\models G_2$

Tell me how

Let us assume  $PA \vdash G_2$ .

$\text{Prf}(\bar{m}, \ulcorner G_2 \urcorner)$ , for some  $m \in \mathbb{N}$

$\text{Prf}(\bar{m}, \text{diag}(\ulcorner \bar{v} \urcorner)) \rightarrow \text{true}$

$PA \vdash \forall x \neg \text{Prf}(x, \text{diag}(\ulcorner \bar{v} \urcorner))$

$PA \vdash \neg \text{Prf}(\bar{m}, \text{diag}(\ulcorner \bar{v} \urcorner))$

$\rightarrow \Sigma_1$ -sentence

$PA \vdash \text{Prf}(\bar{m}, \text{diag}(\ulcorner \bar{v} \urcorner))$   
(By  $\Sigma_1$ -completeness)

$PA \vdash \perp \rightarrow \leftarrow$

# Syntactic Argument (Ch 2.1)

If PA is  $\omega$ -consistent then  $PA \not\vdash \neg \mathcal{G}$

- An arithmetic theory  $T$  is  $\omega$ -inconsistent iff, for some open wff  $\varphi(x)$ ,  $T \vdash \exists x \varphi(x)$ , but for each  $m \in \mathbb{N}$ , we have  $T \vdash \neg \varphi(\bar{m})$ .
- $\omega$ -consistency implies consistency.  $PA + \neg \mathcal{G}$  is consistent, but  $\omega$ -inconsistent. (pp.175)

Proof:  $PA^\dagger = PA + \neg \mathcal{G}$ . If PA is consistent then  $PA + \neg \mathcal{G}$

is consistent.  $\left| \begin{array}{l} PA + \neg \mathcal{G} \vdash \perp, \quad PA \vdash \neg \mathcal{G} \rightarrow \perp \end{array} \right.$

$PA + \neg \mathcal{G} \vdash \exists n \text{Prf}(x, \text{diag}(\ulcorner \mathcal{U} \urcorner))$

$\neg \text{Prf}(\bar{m}, \text{diag}(\ulcorner \mathcal{U} \urcorner))$  holds for  
every  $m \in \mathbb{N}$

$PA + \neg \mathcal{G} \vdash \neg \text{Prf}(\bar{m}, \text{diag}(\ulcorner \mathcal{U} \urcorner))$ , for every  $m \in \mathbb{N}$ .



# Syntactic Argument (Ch 21)

If PA is  $\omega$ -consistent then  $PA \nvdash \neg \mathcal{G}$

$\omega$ -consistency

$\Rightarrow$  consistency

Suppose  $PA \vdash \neg \mathcal{G}$

Tell me how

$PA \vdash \exists y \text{Prf}(y, \text{diag}(\ulcorner \mathcal{G} \urcorner))$

(By consistency  $PA \nvdash \mathcal{G}$ )  $\Rightarrow$  for all  $m \in \mathbb{N}$ ,

$\neg \text{Prf}(\bar{m}, \ulcorner \mathcal{G} \urcorner) \Leftrightarrow \neg \text{Prf}(\bar{m}, \text{diag}(\ulcorner \mathcal{G} \urcorner))$

$\Sigma_1$ -formula

But PA is  $\Sigma_1$ -complete, then

for any  $m \in \mathbb{N}$ .

$PA \vdash \neg \text{Prf}(\bar{m}, \text{diag}(\ulcorner \mathcal{G} \urcorner))$

$\omega$ -inconsistent!!

$\rightarrow \leftarrow$

# Nice theories exhibit incompleteness

(Ch-22)

The lore of p.r. adequacy, p.r. axiomatized and  $\mathcal{L}_{PA}$

$PA,$   
 $ZFC$

If  $T$  is a p.r. adequate, p.r. axiomatized theory whose language includes  $\mathcal{L}_{PA}$ , then there is an  $\mathcal{L}_{PA}$ -sentence  $\varphi$  of Goldbach type such that, if  $T$  is consistent then  $T \nvdash \varphi$ , and

if  $T$  is  $\omega$ -consistent then  $T \nvdash \neg \varphi$ .

$\rightarrow \Pi_1$

$$\boxed{\forall x (x + 0 = x)}$$

$\downarrow$

$Axiom_1(n)$

which is p.r.

Nice theory :  $\left\{ \begin{array}{l} \text{p.r. adequate} \\ \text{p.r. axiomatized} \\ \text{include } \mathcal{L}_{PA}. \end{array} \right.$

If you do a Rosser-Gödel  $Prov_R(m, n)$  then you can switch  $\omega$ -consistency for consistency

# Conclusion to

(ch-23,24)

## Formal study of introspection

$$\begin{aligned} U(y) &:= \neg \exists x \exists z (Prf(x, z) \wedge Diag(y, z)) \\ &\leftrightarrow \forall x \forall z (Diag(y, z) \rightarrow \neg Prf(x, z)) \\ &\leftrightarrow \forall z (Diag(y, z) \rightarrow \neg Prove(z)) \end{aligned}$$

$$(1) PA \vdash G_2 \leftrightarrow \neg Prove(\ulcorner G_2 \urcorner)$$

$$PA \vdash G_2 \leftrightarrow U(\ulcorner G_2 \urcorner)$$

$$PA \vdash G_2 \leftrightarrow \forall z (Diag(\ulcorner G_2 \urcorner, z) \rightarrow \neg Prove(z))$$

$$PA \vdash \forall z (Diag(\ulcorner G_2 \urcorner, z) \leftrightarrow z = \ulcorner G_2 \urcorner) \quad \left[ \begin{array}{l} \text{Since PA can} \\ \text{capture diag}(\cdot) \\ \text{and } diag(\ulcorner G_2 \urcorner) \\ = \ulcorner G_2 \urcorner \end{array} \right]$$

$$PA \vdash G_2 \leftrightarrow \forall z (z = \ulcorner G_2 \urcorner \leftrightarrow \neg Prove(z)) \rightarrow \neg Prove(\ulcorner G_2 \urcorner)$$

$$PA \vdash G_2 \leftrightarrow \neg Prove(\ulcorner G_2 \urcorner)$$

□

$$(2) PA \not\vdash \neg Prove(\ulcorner \perp \urcorner) \quad [\text{Second Incompleteness theorem}]$$

Few ending remarks:

- By Completeness theorem and first incompleteness theorem, We have,  $PA \cup \{\neg G\}$  is satisfiable.  
Hence that model becomes 'non-standard' model.

- More about it can be found in:

"Computability and Logic" by Boolos et al.  
Ch - 25

- Look at Ch - 283 of 'The Logic of Provability' by Boolos  
see a bi-di-erent treatment of Incompleteness theorems.