

06.10.2026

Modal Logic

Fix a countable set of propositional variables or, atomic propositions $P = \{p_1, p_2, \dots\}$.

$\varphi, \psi := \perp \mid \top \mid p \mid \neg \varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi \mid \varphi \leftrightarrow \psi \mid$
 $\Box \varphi \mid \Diamond \varphi$

How do we interpret these formulas?

There are various ways to interpret these formulas.

- * Topological semantics
- * Neighbourhood semantics
- * Algebraic semantics
- * Co-algebraic semantics
- * Kripke semantics.

Kripke Frame :

A Kripke frame is a pair $F = (W, R)$, where W is a non-empty set, and $R \subseteq W \times W$ is a binary relation.

Kripke Model :

A Kripke model is a pair $M = (F, V)$, where $F = (W, R)$ is a Kripke frame and $V : W \rightarrow 2^P$ is a valuation.

tion function. We write: $M = (W, R, V)$.

- We refer to W as a set of (possible) worlds.
- When $p \in V(w)$, we say p holds at w in M , otherwise p is false at w in M .
- Given a model M , and a world w in M , (M, w) is termed as a **pointed model**.

Truth definition

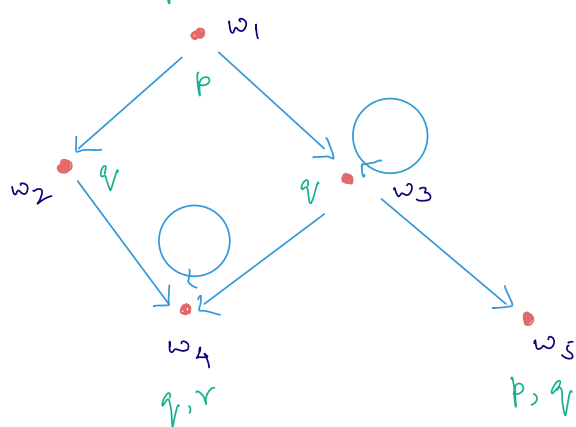
The notion ' φ holds at the world w in M ', denoted by $M, w \models \varphi$ is defined as follows:

- $M, w \models \perp$, never
- $M, w \models \top$, always
- $M, w \models p$ iff $p \in V(w)$
- $M, w \models \neg \varphi$ iff $M, w \not\models \varphi$
- $M, w \models \varphi \wedge \psi$ iff $(M, w \models \varphi \text{ and } M, w \models \psi)$
- $M, w \models \varphi \vee \psi$ iff $(M, w \models \varphi \text{ or } M, w \models \psi)$
- $M, w \models \varphi \rightarrow \psi$ iff $(M, w \models \varphi \text{ implies } M, w \models \psi)$
- $M, w \models \varphi \leftrightarrow \psi$ iff $(M, w \models \varphi \text{ iff } M, w \models \psi)$
- $M, w \models \Box \varphi$ iff **for all** $w' \in W$ with $w R w'$, $M, w' \models \varphi$
- $M, w \models \Diamond \varphi$ iff **there exists** $w' \in W$ such that $w R w'$ and $M, w' \models \varphi$.

* φ is **satisfiable** if there is a model $\mathcal{M} = (W, R, V)$ and some $w \in W$, such that $\mathcal{M}, w \models \varphi$.

* φ is **valid** if $\neg \varphi$ is not satisfiable.

Example



$$\mathcal{M} = (W, R, V)$$

$$W = \{w_1, w_2, \dots, w_5\}$$

$$R = \{(w_1, w_2), (w_1, w_3), \dots, (w_3, w_5)\}$$

$$V(w_1) = \{p\}, \dots, V(w_5) = \{p, q\}$$

$$\mathcal{M}, w_4 \models q \wedge r ; \quad \mathcal{M}, w_2 \models \Box q ; \quad \mathcal{M}, w_3 \models \Diamond r ;$$

$$\mathcal{M}, w_3 \not\models \Box r ; \quad \mathcal{M}, w_1 \models \Diamond \Box q ; \quad \mathcal{M}, w_5 \models \Box r$$

Remark: $\Box$ -formulas are always satisfied in states w where, $\{w' : (w, w') \in R\} = \emptyset$.

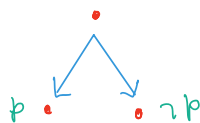
Examples of valid formulas

$$- \Box(p \vee \neg p)$$

$$- \Box p \vee \Box \neg p$$

$$- \Box \neg p \rightarrow \neg \Box p$$

$$- \Diamond(p \wedge q) \rightarrow (\Diamond p \wedge \Diamond q)$$



$$(W, R, V)$$

$w \in W$

}

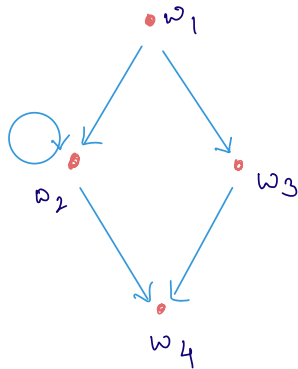
H.W.

$$- (\Diamond p \wedge \Diamond q) \rightarrow \Diamond(p \wedge q)$$

$$\Box \perp$$

$$\Diamond T$$

Defining worlds in a model



$$\mathcal{M}, w_4 \models \Box \perp$$

$$\mathcal{M}, w_3 \models \Diamond \Box \perp \wedge \Box \Box \perp$$

$$\mathcal{M}, w_2 \models \Diamond \Box \perp \wedge \Diamond \Diamond T$$

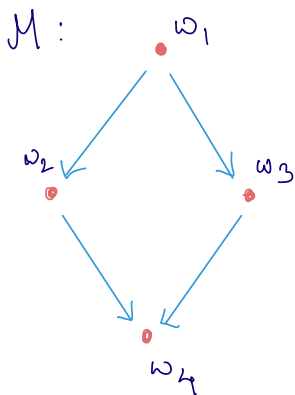
$$\mathcal{M}, w_1 \models \Diamond(\Diamond \Box \perp \wedge \Box \Box \perp)$$

$$\Diamond T$$

$$\underbrace{\Diamond \dots \Diamond}_n T$$

A world w in a model $\mathcal{M}$ is said to be **definable in modal logic** if there is a formula φ such that $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w' \not\models \varphi$ for all $w' \in W_{\mathcal{M}} \setminus \{w\}$.

From definability to (in)distinguishability.

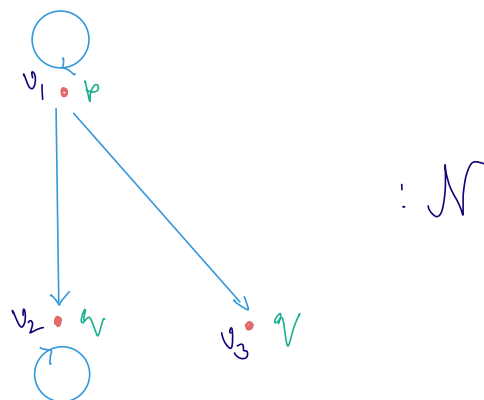
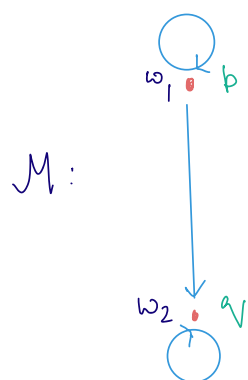


Is it possible to distinguish w_2 and w_3 in modal logic?

$$\mathcal{M}, w_2 \models \varphi \ \& \ \mathcal{M}, w_3 \not\models \varphi$$

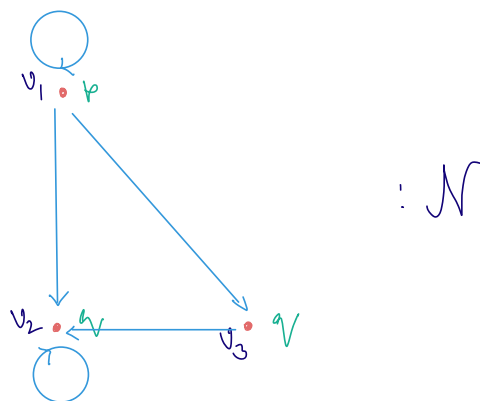
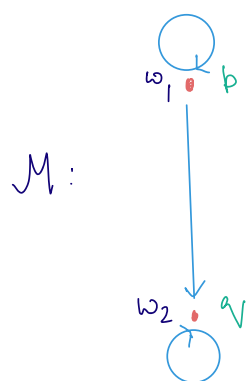
H.W. Show that $(\mathcal{M}, w_2)$ and $(\mathcal{M}, w_3)$ satisfy the same modal formulas.

We now consider the notion of distinguishing worlds in distinct models.



Can you distinguish between (M, w_1) and (N, v_1) by a modal formula? Yes. $\Diamond \Box \perp$, $\Box \Diamond q$

Consider the following two models.



Can you distinguish between (M, w_1) and (N, v_1) now?

Actually, no modal formulas can distinguish between (M, w_1) and (N, v_1) .

How do we prove this?

Modal equivalence

Two pointed models (M, w) and (N, v) are said to be modally equivalent if for all modal formulas φ , $M, w \models \varphi$ iff $N, v \models \varphi$, that is, they satisfy the same modal formulas.

Can we consider a notion of invariance between the pointed models to capture modal equivalence?

Bisimulation

Let $M_1: (W_1, R_1, V_1)$ and $M_2: (W_2, R_2, V_2)$ be two models. Let $w_1 \in W_1$, $w_2 \in W_2$. (M_1, w_1) is said to be bisimilar to (M_2, w_2) ($(M_1, w_1) \cong (M_2, w_2)$) if there is a binary relation $Z \subseteq W_1 \times W_2$ such that $(w_1, w_2) \in Z$ and for all $x \in W_1$ and $y \in W_2$, $(x, y) \in Z$ implies:

* for all atomic propositions $p \in P$, $p \in V_1(x)$ iff $p \in V_2(y)$.

(Atomic Harmony)

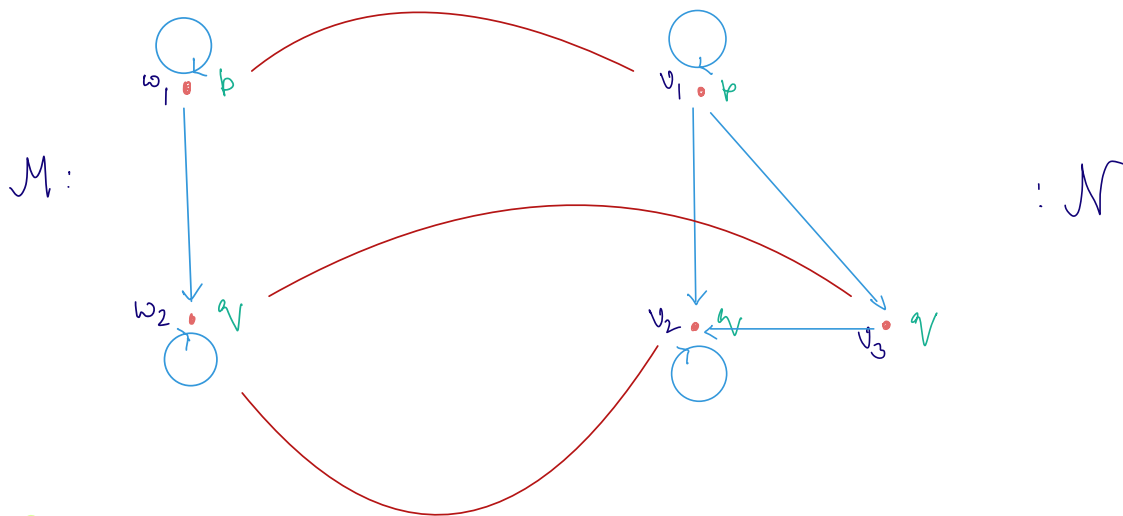
* if $x R_1 x'$, then there is $y' \in W_2$ s.t. $y R_2 y'$ and $x' Z y'$.

(Zig)

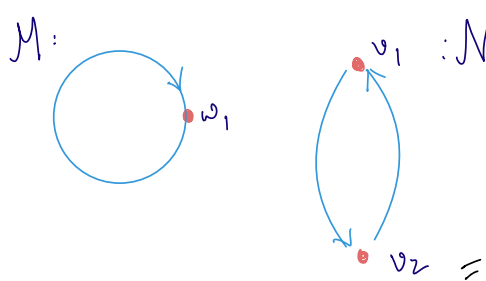
* if $y R_2 y'$, then there is $x' \in W_1$ s.t. $x R_1 x'$ and $x' Z y'$.

(Zag)

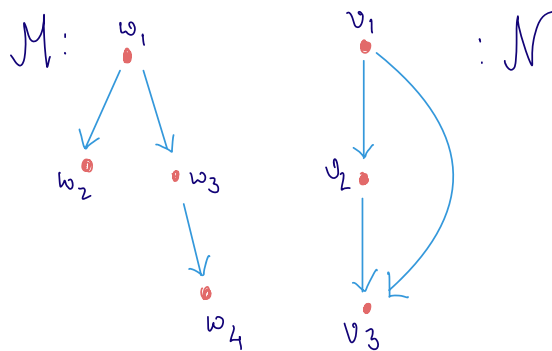
Examples



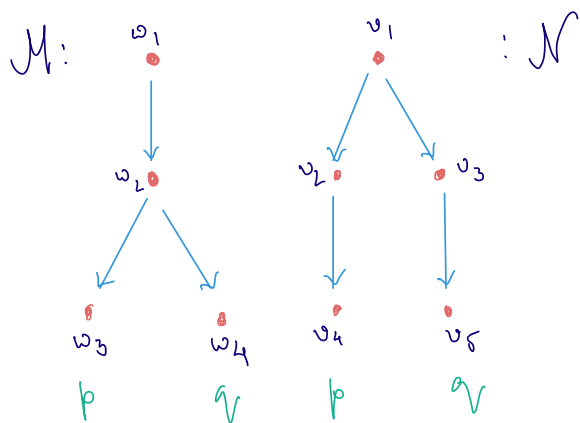
H.W. Check whether the following pairs of pointed models are bisimilar.



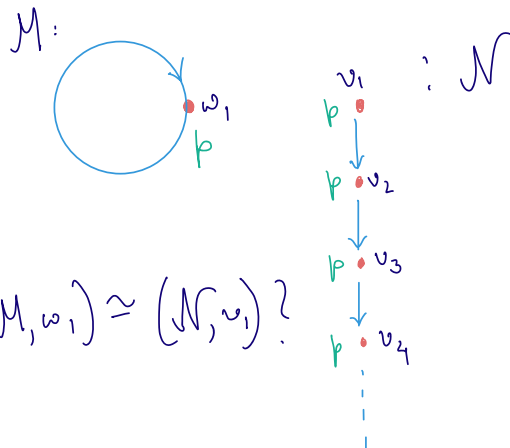
$$(M, w_1) \simeq (N, v_1) ?$$



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Invariance Lemma: Let (M, s) and (N, t) be two pointed Kripke models with $(M, s) \cong (N, t)$. Then, they satisfy the same modal formulas.

Proof: Let $(M, s) \cong (N, t)$. To show that $M, s \models \varphi$ iff $N, t \models \varphi$ for all modal formulas φ . We prove this by applying induction on the size of φ .

Base Case: $\varphi := p$. Then, $M, s \models p$ iff $N, t \models p$ by the atomic harmony condition.

Induction Hypothesis: Suppose the result holds for all formulas of size $\leq n$.

Induction Step: Let the size of φ be $n+1$.

Case 1: $\varphi := \neg \psi$. $M, s \models \varphi$ iff $M, s \models \neg \psi$ iff $M, s \not\models \psi$ iff $N, t \not\models \psi$ (I.H.) iff $N, t \models \neg \psi$ iff $N, t \models \varphi$.

Case 2: $\varphi := \psi \vee \chi$. Follows similarly.

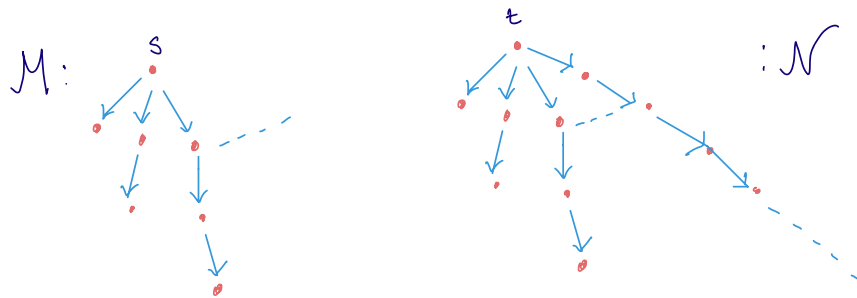
Case 3: $\varphi := \Diamond \psi$. Let $M, s \models \Diamond \psi$. Then, there is s' such that $s R_M s'$ and $M, s' \models \psi$. Since $s R_M s'$ and $(M, s) \cong (N, t)$, there is t' in N such that $t R_N t'$ and $(M, s') \cong (N, t')$ (by the **Zig** condition). Since $M, s' \models \psi$ and $(M, s') \cong (N, t')$, by I.H.,

$\mathcal{N}, t' \models \Psi$ (size of $\Psi \leq m$). Then, $\mathcal{N}, t \models \Diamond \Psi$.

So, $(\mathcal{M}, s) \models \Diamond \Psi$ implies $(\mathcal{N}, t) \models \Diamond \Psi$. The other direction can be shown similarly using the **Tag** condition. This completes the proof.

What about the converse? Does modal equivalence imply bisimulation?

NO!!



H.W. Prove that $(\mathcal{M}, s)$ and $(\mathcal{N}, t)$ are modally equivalent but are not bisimilar.

Modal depth of a formula

Given a modal formula ϕ , the **modal depth** of ϕ ($md(\phi)$) is defined as follows:

$$md(p) = 0; \quad md(\neg\psi) = md(\psi); \quad md(\psi \vee \chi) = md(\psi \wedge \chi) = md(\psi \rightarrow \chi) = md(\psi \leftrightarrow \chi) = \max\{md(\psi), md(\chi)\}; \quad md(\Diamond\psi) = md(\Box\psi) = md(\psi) + 1.$$

Example.

$$md(\Box\Diamond p) = 2; \quad md(\Diamond\Diamond\Box p \vee \Diamond\Box(\Box p \wedge q)) = 3$$

H.W. Show that $\Box\phi \leftrightarrow \neg\Diamond\neg\phi$ is a valid formula.

Can we find a restricted class of models where the converse holds?

Let us consider **finite** models and check whether the converse result holds.

Proposition: Let M, N be finite models such that (M, s) and (N, t) are modally equivalent. Then they are bisimilar.

To show $(M, s) \simeq (N, t)$, we need to find a bisimulation $Z \subseteq W_M \times W_N$ with $(s, t) \in Z$. Let us consider $Z \subseteq W_M \times W_N$ defined as follows: for all $x \in W_M, y \in W_N$, $x Z y$ iff (M, x) and (N, y) satisfy the same modal formulas. Of course, $s Z t$. We just need to show that Z is a bisimulation: let $(x, y) \in Z$.

- **Atomic Harmony:** Follows by the definition of Z .

- **Zig:** Let $x R_M x'$. To show that there is $y' \in W_N$ such that $y R_N y'$ and $x' Z y'$. Suppose not. Let $I = \{u : y R_N u\}$

Can I be empty? **NO**. Since $x R_M x'$, we have $M, x \models \Diamond T$. But $x Z y$ and hence $N, y \models \Diamond T$. So, I is non-empty.

Since N is finite, I is finite - let $I = \{u_1, \dots, u_m\}$.

Then, for all $i \in \{1, \dots, m\}$, it is not the case that $x' Z u_i$.

Then for each i , there is a modal formula ϕ_i , say, such that $M, x' \models \phi_i$ and $N, u_i \not\models \phi_i$. Then, $M, x' \models \phi_1 \wedge \dots \wedge \phi_m$,

where as, for each i , $\mathcal{N}, u_i \not\models \phi_i$. Then we have:
 $\mathcal{M}, x \models \Diamond(\phi_1 \wedge \dots \wedge \phi_m)$ and $\mathcal{N}, y \not\models \Diamond(\phi_1 \wedge \dots \wedge \phi_m)$. So
we have a contradiction as $x \mathbb{Z} y$. Hence the result.

• **Zag**: Can be shown similarly.

This completes the proof.

Remark: For finite models, modal equivalence implies bisimulation.

Can we generalise this notion a bit further? Where did we use the finiteness?

We needed the set I to be finite, which is the image set of the underlying relation. Thus, rather than considering the whole model to be finite, we can consider such image sets to be finite and the proof will go through.

Image-finite model

A model $\mathcal{M} = (W, R, V)$ is said to be an **image-finite** model if for all $w \in W$, $\text{Image}(w) = \{w' \in W : w R w'\}$ is finite.

Proposition: If $\mathcal{M}$ and $\mathcal{N}$ are image-finite models, then $(\mathcal{M}, s)$ and $(\mathcal{N}, t)$ are modally equivalent imply $(\mathcal{M}, s)$ and $(\mathcal{N}, t)$ are bisimilar.