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An application on expressivity

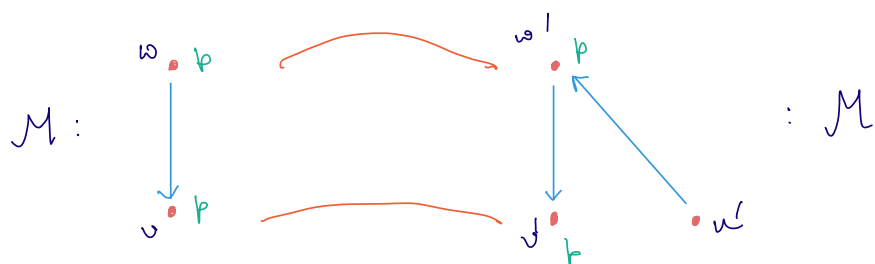
We use the concept of bisimulation to show what modal logic cannot express.

Universal operator $\Box$.

$M, w \models \Box \varphi$ iff for all $v \in W_M$, $M, v \models \varphi$.

Is the universal operator definable in the modal language?

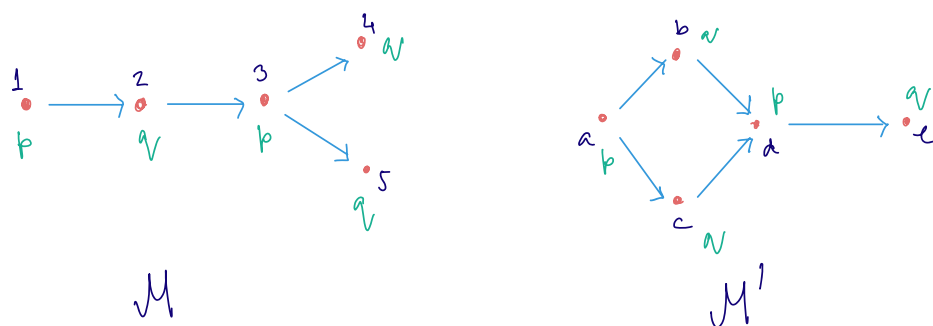
Suppose it is definable. Let $\alpha(\varphi)$ be a modal formula such that for all models M and for all worlds w in M
 $M, w \models \Box \varphi$ iff $M, w \models \alpha(\varphi)$. Now, consider the following:



Clearly, $M, w \models \Box p$, but, $M', w' \not\models \Box p$. So, $M, w \models \alpha(p)$ but $M', w' \not\models \alpha(p)$. However, (M, w) and (M', w') are bisimilar and so, satisfy the same modal formulas. So, we should have had $M, w \models \alpha(p)$ iff $M', w' \models \alpha(p)$, but that is not the case. So, we have a contradiction. Thus, the universal operator $\Box$ is not expressible in the logic.

First-order logic (FOL) vs. Modal logic (ML)

How similar / different they are?



$$(\mathcal{M}, 1) \simeq (\mathcal{M}', a)$$

Thus, these pointed models cannot be distinguished by **model formulas**.

FOL formula: $\exists y_1 \exists y_2 \exists y_3 (\neg(y_1 = y_2) \wedge \neg(y_2 = y_3) \wedge \neg(y_3 = y_1) \wedge Rxy_1 \wedge Rxy_2 \wedge Ry_1y_3 \wedge Ry_2y_3)$

This formula holds in M' but does not hold in M .

So, ML cannot distinguish between these two models, but FOL can. Let us now explore this relationship more formally. To discuss this let us first fix the parameters.

Parameters of the first-order language.

R : a binary relation symbol.

$P_0, P_1, P_2, \dots$: unary relation symbol

Thus, a Kripke model $M: (W, R, V)$ can be seen as an FOL-structure (D, I) as follows: $D = W$; $I(R) = R$; $I(P_i) = V(p_i)$.

Standard translation

A translation of ML formulas into FOL formulas is given as follows: Consider a variable x to provide

the translation, known as the **standard translation**
(ST_x): $ST_x(p_i) = P_i x$.

$$ST_x(\perp) = \neg(x \equiv x)$$

$$ST_x(\neg \phi) = \neg ST_x(\phi)$$

$$ST_x(\phi \vee \psi) = ST_x(\phi) \vee ST_x(\psi)$$

$$ST_x(\phi \wedge \psi) = ST_x(\phi) \wedge ST_x(\psi)$$

$$ST_x(\phi \rightarrow \psi) = ST_x(\phi) \rightarrow ST_x(\psi)$$

$$ST_x(\phi \leftrightarrow \psi) = ST_x(\phi) \leftrightarrow ST_x(\psi)$$

$$ST_x(\Diamond \phi) = \exists y (Rxy \wedge ST_y(\phi))$$

$$ST_x(\Box \phi) = \forall y (Rxy \rightarrow ST_y(\phi))$$

Find: $ST_x(\Diamond(\Box p \rightarrow q))$

Answer: $\exists y (Rxy \wedge (\forall z (Ryz \rightarrow Pz) \rightarrow Qy))$

Some related results:

1. For all Kripke models M , and all worlds w in M ,
 $M, w \models \phi$ iff $M[x \mapsto w] \models ST_x(\phi)$ for all modal formulas ϕ .
2. For all Kripke models M , $M \models \phi$ ($M, w \models \phi$ for all w in M)
iff $M \models \forall x ST_x(\phi)$, for all modal formulas ϕ .

Proof of 1: We prove this by induction on the size of the formula ϕ .

Base case: $\phi := p$. Then, $M, w \models p$ iff $w \in V(p)$ iff $w \in I(p)$
iff $M[x \mapsto w] \models P_x$ iff $M[x \mapsto w] \models ST_x(p)$.

Induction Hypothesis: Suppose the result holds for all formulas of size $\leq n$.

Induction Step: Let φ be a formula of size $m+1$.

Case 1: $\varphi := \neg \psi$. Then, $M, w \models \varphi$ iff $M, w \models \neg \psi$ iff $M, w \not\models \psi$ iff $M_{[x \rightarrow w]} \not\models ST_x(\psi)$ iff $M_{[x \rightarrow w]} \models \neg ST_x(\psi)$ iff $M_{[x \rightarrow w]} \models ST_x(\neg \psi)$

Case 2: $\varphi := \psi \vee \chi$. The proof is similar.

Case 3: $\varphi := \Diamond \psi$. Suppose $M, w \models \Diamond \psi$. Then, there exists v in M s.t. $w R v$ and $M, v \models \psi$. Since $w R v$, we have:

$M_{[x \rightarrow w, z \rightarrow v]} \models R_x z$. Since, $M, v \models \psi$, we have by I.H.:

$M_{[x \rightarrow v]} \models ST_x(\psi)$. Let us consider a new variable y .

Then, we have: $M_{[y \rightarrow v]} \models ST_y(\psi)$ and $M_{[x \rightarrow w, y \rightarrow v]} \models R_{xy}$.

Now, as x does not occur free in $ST_y(\psi)$, we have:

$M_{[x \rightarrow w, y \rightarrow v]} \models ST_y(\psi)$. So, $M_{[x \rightarrow w, y \rightarrow v]} \models R_{xy} \wedge ST_y(\psi)$

Then, $M_{[x \rightarrow w]} \models \exists y (R_{xy} \wedge ST_y(\psi))$, that is,

$M_{[x \rightarrow w]} \models ST_x(\Diamond \psi)$.

Conversely, suppose that $M_{[x \rightarrow w]} \models ST_x(\Diamond \psi)$. So, $M_{[x \rightarrow w]} \models \exists y (R_{xy} \wedge ST_y(\psi))$. Then, for some v in M , $M_{[x \rightarrow w, y \rightarrow v]} \models R_{xy} \wedge ST_y(\psi)$.

Thus, $M_{[x \rightarrow w, y \rightarrow v]} \models R_{xy}$ and $M_{[x \rightarrow w, y \rightarrow v]} \models ST_y(\psi)$.

From the first condition, we have $w R v$. From the second one, since x does not occur free in $ST_y(\psi)$,

$M_{[y \rightarrow v]} \models ST_y(\psi)$. Then, by I.H., $M, v \models \psi$. So,

as $w R v$ and $M, v \models \psi$, $M, w \models \Diamond \psi$. This completes the proof.

Proof of 2 : H.W.

Applications

- Compactness Theorem: An infinite set of modal formulas is satisfiable iff every finite subset of it is satisfiable.

H.W. Prove this result using standard translation.

- Lowenheim-Skolem Theorem: If a set of modal formulas is satisfiable in an infinite model then it is satisfiable in models of every infinite cardinality.

H.W. Prove this result using standard translation.

So any modal logic formula is expressible in FOL with relevant parameters, but not conversely.

Why is the above statement true? H.W.

Can we think of a collection of FOL formulas which are equivalent to modal formulas?

We explore such a direction below.

Translation in FO_2

Proposition: Any modal logic formula can be translated to a first-order logic formula containing at most two variables (FO_2)

Proof idea: Rename variables.

$$ST_x(p) = P_x$$

$$ST_y(p) = P_y$$

$$ST_x(\perp) = \neg(x = x)$$

$$ST_y(\perp) = \neg(y = y)$$

$$ST_x(\neg\phi) = \neg ST_x(\phi)$$

$$ST_y(\neg\phi) = \neg ST_y(\phi)$$

$$ST_x(\phi \vee \psi) = ST_x(\phi) \vee ST_x(\psi)$$

$$ST_y(\phi \vee \psi) = ST_y(\phi) \vee ST_y(\psi)$$

$$ST_x(\Diamond\psi) = \exists y (R_{xy} \wedge ST_y(\psi))$$

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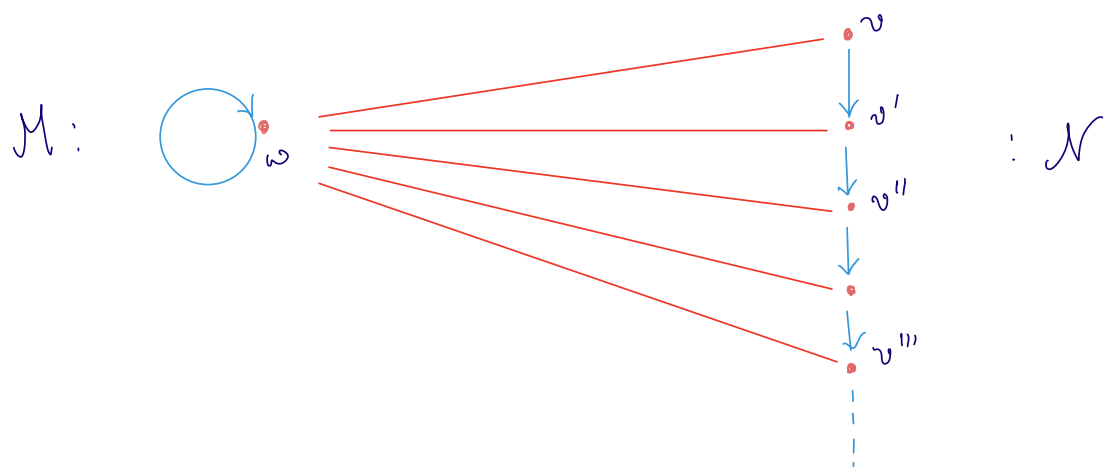
Example

$$\phi := \Diamond(\Box p \rightarrow q)$$

$$ST_x(\phi) = \exists y (R_{xy} \wedge (\forall x (R_{yx} \rightarrow P_x) \rightarrow Q_y))$$

What about the converse? Can every first-order formula in two variables be equivalent to the translation of a modal formula?

NO. Consider $\phi := R_{xx}$. Suppose ψ is a modal formula s.t. $ST_x(\psi)$ is equivalent to ϕ . Let us consider the following models.



We have : $M_{[x \rightarrow w]} \models \varphi$ and $N_{[x \rightarrow v]} \not\models \varphi$. But , $M, w \cong N, v$, and so they satisfy the same modal formulas. Thus we have a contradiction. So, there is no modal formula φ with $ST_2(\varphi)$ equivalent to φ .

Can we characterize some set of first-order formulas that are equivalent to the translation of modal formulas ?

Yes. Those first-order formulas that are invariant under bisimulation : A first-order formula $\varphi(x)$ is said to be **invariant under bisimulation** if for all Kripke models M and N , for all worlds w in M and v in N and all bisimulations Z between M and N such that $w Z v$, we have :

$$M_{[x \rightarrow w]} \models \varphi(x) \text{ iff } N_{[x \rightarrow v]} \models \varphi(x).$$

van Benthem characterization theorem : Let $\varphi(x)$ be a first-order formula with one free-variable x . Then $\varphi(x)$ is invariant under bisimulation iff $\varphi(x)$ is logically equivalent to a standard translation of a modal formula.

The proof of the 'if' part is trivial. The proof of the 'only-if' part is quite involved, and we leave it for a future discussion.

Finite model property (FMP).

A logic L is said to have FMP if any satisfiable formula φ in L is satisfiable in a finite model.

Does FOL have FMP?

H.W. Provide an answer to this question.

Does ML have FMP?

Yes. In what follows we will prove this result. We will prove a stronger result.

Strong finite model property (SFMP)

A logic L is said to satisfy **SFMP** if any satisfiable formula ϕ in L is satisfiable in a finite model of size at most $2^{|\phi|}$, where $|\phi|$ denotes the number of subformulas in ϕ .

Proposition: **ML has SFMP.**

Proof: Let ϕ be a modal formula. Let $\text{Sub}(\phi)$ denote the set of subformulas of ϕ . Then $\text{Sub}(\phi)$ is a finite set. Assume that ϕ is satisfiable. Then, there is a model $M: (W, R, V)$ and a world w in M , s.t. $M, w \models \phi$. We have to reduce the size of W to a finite set while preserving satisfaction. To do this, we define a relation $\sim_\phi$ on W as follows: For all $w, v \in W$, $w \sim_\phi v$ iff for all $\psi \in \text{Sub}(\phi)$, $M, w \models \psi$ iff $M, v \models \psi$.

Is $\sim_\phi$ an equivalence relation? **Check!**

Then, $\sim_\phi$ partitions W into equivalence classes. Let $W_\sim = \{[w] : w \in W\}$. What is $|W_\sim|$?

Let us define a map $g: W_\sim \rightarrow \mathcal{P}(\text{Sub}(\phi))$:

$g([w]) = \{\psi \in \text{Sub}(\phi) : M, w \models \psi\}$. So, $g([w]) \subseteq \text{Sub}(\phi)$

Is g well-defined? We have to show that if $w \sim_\phi v$,

$g([w]) = g([v])$. Suppose $w \sim_\phi v$. Then $M, w \models \psi$ iff

$M, v \models \psi$ for all $\psi \in \text{Sub}(\phi)$. So, $g([w]) = g([v])$.

Is g injective? Let $w, v \in W$, with $g([w]) = g([v])$.

Then $\{\psi \in \text{Sub}(\phi) : M, w \models \psi\} = \{\psi \in \text{Sub}(\phi) : M, v \models \psi\}$.

So, $w \sim_{\phi} v$ and hence, $[w] = [v]$. Thus g is an injective map from $W_{\sim}$ to $\mathcal{P}(\text{Sub}(\phi))$. Since,

$|\mathcal{P}(\text{Sub}(\phi))| = 2^{|\phi|}$, we have: $|W_{\sim}| \leq 2^{|\phi|}$. Thus, starting from W , we have a finite bounded set $W_{\sim}$.

To get a model, we need to find $R_{\sim}$ and $V_{\sim}$ on $W_{\sim}$.

Let us define $V_{\sim}(p) = \{[w] : w \in V(p)\}$; for all propositional atoms p . How do we define $R_{\sim}$?

Let us postpone this discussion and consider $M_{\sim} = (W_{\sim}, R_{\sim}, V_{\sim})$.

We now show the following lemma.

Lemma: For all formulas $\psi \in \text{Sub}(\phi)$, and for all worlds w in M , $M, w \models \psi$ iff $M_{\sim}, [w] \models \psi$.

Proof: We prove this by applying induction on the size of ψ .

Base case: $\psi := p$. $M, w \models p$ iff $w \in V(p)$ iff $[w] \in V_{\sim}(p)$ iff $M_{\sim}, [w] \models p$.

Induction Hypothesis: Suppose the result holds for all formulas of size $\leq n$.

Induction Step: Let the size of ϕ be $n+1$.

Case 1: $\phi := \neg \psi$
Case 2: $\phi := \psi \vee \chi$ } Proof is as usual.