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Bound variables : Variables that occur under the scope of a quantifier.
Free variables : Variables that are not bound.

- $\forall x \exists y \, P_1^2 xy$: Both x and y are bound.
- $\forall x \, P_1^2 xy$: x is bound, y is free.

Set of free variables occuring in terms

- $FV(x) = \{x\}$
- $FV(c) = \emptyset$
- $FV(f_i^n t_1 \cdots t_n) = \bigcup_{k=1}^{n} FV(t_k)$

$FV(\neg \varphi) =$

Set of free variable occuring in formulas

- $FV(t_1 \equiv t_2) = FV(t_1) \cup FV(t_2)$
- $FV(p_i^n t_1 \cdots t_n) = \bigcup_{k=1}^{n} FV(t_k)$
- $FV(\neg \varphi) = FV(\varphi)$
- $FV(\varphi \vee \psi) = FV(\varphi \wedge \psi) = FV(\varphi \to \psi)$

$= FV(\varphi \leftrightarrow \psi) = FV(\varphi) \cup FV(\psi)$

- $FV(\forall x \varphi) = FV(\varphi) \setminus \{x\}$.
- $FV(\exists x \varphi) = FV(\varphi) \setminus \{x\}$.

Proposition: Let φ be a formula and (D, I) be a structure. For any two assignment functions g_1, g_2, if g_1 and g_2 agrees on $FV(\varphi)$, we have: $(D, I, g_1) \models \varphi$ iff $(D, I, g_2) \models \varphi$.

Size of a formula: The number of connectives and quantifiers present in the formula.

Proof: We prove this by applying induction on the size of the formula φ. To prove:

$M_1 : (D, I, g_1) \models \varphi$ iff $M_2 : (D, I, g_2) \models \varphi$

Base case: (i) $\varphi : t_1 \equiv t_2$: Then, $M_1 \models \varphi$ iff $M_1 \models t_1 \equiv t_2$ iff $g_1(t_1) = g_1(t_2)$ iff $g_2(t_1) = g_2(t_2)$ iff $M_2 \models t_1 \equiv t_2$ iff $M_2 \models \varphi$. Why?

(ii) $\varphi : p_i^n t_1 \cdots t_n$: Then, $M_1 \models \varphi$ iff

$\mathcal{M}_1 \models p_i^n t_1 \cdots t_n$ iff $(g_1(t_1), \cdots, g_1(t_n)) \in I(p_i^n)$

$(g_2(t_1), \cdots, g_2(t_n)) \in I(p_i^n)$ iff $\mathcal{M}_2 \models p_i^n t_1 \cdots t_n$ iff

$\mathcal{M}_2 \models \varphi$

Induction Hypothesis : Suppose the result holds for all formulas of size $\leq n$.

Induction Step : Let the size of φ be $n+1$.

$\boxed{\text{Case I}}$ $\varphi = \neg\psi$: $\mathcal{M}_1 \models \varphi$ iff $\mathcal{M}_1 \models \neg\psi$ iff

$\mathcal{M}_1 \not\models \psi$ iff $\mathcal{M}_2 \not\models \psi$ (by I.H.) iff $\mathcal{M}_2 \models \neg\psi$

iff $\mathcal{M}_2 \models \varphi$.

$\boxed{\text{Case II}}$ $\varphi = \psi \wedge \chi$: $\mathcal{M}_1 \models \varphi$ iff $\mathcal{M}_1 \models \psi \wedge \chi$

iff $\mathcal{M}_1 \models \psi$ and $\mathcal{M}_1 \models \chi$ iff $\mathcal{M}_2 \models \psi$ and $\mathcal{M}_2 \models \chi$

(by I.H.) iff $\mathcal{M}_2 \models \psi \wedge \chi$ iff $\mathcal{M}_2 \models \varphi$.

Remark : It is enough to show for the connectives $\neg$ and $\wedge$. We do not need to show the cases for $\vee$, $\rightarrow$ and $\leftrightarrow$ separately.

$\boxed{\text{Case III}}$ $\varphi := \forall x\, \psi$: $\mathcal{M}_1 \models \varphi$ iff $\mathcal{M}_1 \models \forall x\, \psi$ iff for all $d \in D$, $\mathcal{M}_{1[x \to d]} \models \psi$ iff for all $d \in D$ $\mathcal{M}_{2[x \to d]} \models \psi$ iff $\mathcal{M}_2 \models \forall x\, \psi$ iff $\mathcal{M}_2 \models \varphi$.

To prove iff : Take any $d \in D$. Then we have :

$$\mathcal{Y}_{i[x \to d]}(y) = \begin{cases} \mathcal{Y}_i(y), & y \neq x \\ d, & y = x \end{cases}$$

for $i = 1, 2$.

Also, $FV(\varphi) = FV(\psi) \setminus \{x\}$

We will be able to prove iff if we can show that $\mathcal{Y}_{1[x \to d]}$ and $\mathcal{Y}_{2[x \to d]}$ argue on $FV(\psi)$.

Because then, by I.H., we will have :

$$\boxed{\mathcal{M}_{1[x \to d]} \models \psi \text{ iff } \mathcal{M}_{2[x \to d]} \models \psi}$$

Now, $\mathcal{Y}_{1[x \to d]}$ and $\mathcal{Y}_{2[x \to d]}$ agree on $FV(\varphi)$ and they also agree on x $(FV(\varphi) = FV(\psi) \setminus \{x\})$

Thus : $\mathcal{M}_{1[x \to d]} \models \psi$ iff $\mathcal{M}_{2[x \to d]} \models \psi$.

Since we have taken an arbitrary $d \in D$, we have that for all $d \in D$, $M_{1[x \to d]} \models \psi$ iff $M_{2[x \to d]} \models \psi$. Thus $M_1 \models \forall x \psi$ iff $M_2 \models \forall x \psi$, that is, $M_1 \models \varphi$ iff $M_2 \models \varphi$. This completes the proof.

Why did not we do the case for $\exists x \varphi$?

Sentences

A formula which does not have any free variable is said to be a sentence.

Example: $\forall x (p_1^1 x)$, $\forall x \exists y (x \equiv y)$

Corollary: Let φ be a sentence and (D, I) be a structure. Then, either $(D, I, \mathcal{G}) \models \varphi$ for all assignment functions $\mathcal{G}$ or, $(D, I, \mathcal{G}) \not\models \varphi$ for all assignment functions $\mathcal{G}$.

H. W. Prove this corollary.

Remark: When φ is a sentence, we have: φ is either true in a structure (D, I), $((D, I) \models \varphi)$ or, φ is false in the structure $((D, I) \not\models \varphi)$.

Expressivity of first-order language.

Consider a first-order language $\mathcal{L}$ s.t. $C_{\mathcal{L}} = \mathcal{F}_{\mathcal{L}} = \mathcal{P}_{\mathcal{L}} = \Phi$. For this discussion let us assume that D can be empty as well. Using this language, we can at least talk about sets (as we cannot put any structure on these sets).

- $\forall x \, \neg(x \equiv x)$: empty set
- $\forall x \forall y (x \equiv y)$: sets with ≤ 1 element
- $\exists x \forall y (x \equiv y)$: singleton sets
- $\exists x \exists y (\neg(x \equiv y) \wedge \forall z (z \equiv x \vee z \equiv y))$: sets with exactly two elements
- $\exists x \exists y \exists z (\neg(x \equiv y) \wedge \neg(y \equiv z) \wedge \neg(z \equiv x) \wedge \forall w (w \equiv x \vee w \equiv y \vee w \equiv z))$: sets with exactly three elements

Similarly, we can express sets having exactly k elements, for any finite number k.

Can we express sets containing infinitely many elements? (Think about it!)

Definable relations in structures.

Let $\mathcal{L}$ be a first-order language and let $\mathcal{A}$ be an $\mathcal{L}$-structure $(\mathcal{A} : (D, I))$. An n-ary relation R on D is said to be definable in $\mathcal{A}$ if there is an $\mathcal{L}$-formula φ, say, whose free variables are $x_1, x_2, \cdots, x_n$, such that:

$$(a_1, \cdots, a_n) \in R \text{ iff } \mathcal{A}_{[x_1 \to a_1, x_2 \to a_2, \cdots, x_n \to a_n]} \models \varphi \,/\, \varphi(x_1, x_2, \cdots, x_n)$$

for all $a_1, a_2, \cdots, a_n \in D$.

Example

1. $\mathcal{R} : (\mathbb{R}, 0, 1, +, \cdot)$ $\qquad \mathcal{L}_{\mathcal{R}} = \{\{0,1\}, \{+, \cdot\}, \emptyset\}$

(a) $P = \{a : a \geq 0\}$ $\qquad \varphi(x) := \exists y \, (x = y \cdot y)$

(b) $\leq \; = \{(a,b) : a \leq b\}$ $\qquad \varphi(x,y) := \exists z \, (y = x + z \cdot z)$

2. $G = (\{a, b, c\}, \{(a,b), (a,c)\})$ $\qquad \mathcal{L}_G = \{\emptyset, \emptyset, \{E\}\}$

$b \leftarrow a \rightarrow c$ $\qquad$ edge relation : E

(a) $\{b, c\}$ $\qquad \varphi(x) := \exists y \, E y x$

(b) $\{b\}$ $\qquad$ Not definable (no way to distinguish $\{b\}$ and $\{c\}$)

Definability of a class of structures:

Let $\mathcal{K}$ be a class of structures. $\mathcal{K}$ is said to be definable in an FO language if there is a sentence σ, say, such that

$$\mathcal{K} = \{ S : S \models \sigma \} \quad (\mathrm{Mod}(\sigma))$$

Example

$\mathcal{K}$ = Class of all groups ; $\mathcal{L} = \{ e, * \}$

$$\sigma := \forall x \forall y \forall z \, ((x * y) * z \equiv x * (y * z))$$
$$\wedge \ \forall x \, ((x * e) \equiv x)$$
$$\wedge \ \forall x \, ((e * x) \equiv x)$$
$$\wedge \ \forall x \exists y \, ((x * y) \equiv e)$$
$$\wedge \ \forall x \exists y \, ((y * x) \equiv e)$$

To understand what is definable and what is not, let us first discuss distinct structures satisfying the same FO-formulas.

Two models $M_1 = (D_1, I_1, g_1)$ and $M_2 = (D_2, I_2, g_2)$

are fo-equivalent, if for all fo-formulas φ, $\mathcal{M}_1 \models \varphi$ iff $\mathcal{M}_2 \models \varphi$.

Elementary equivalence

Two structures $\mathcal{A} = (D_A, I_A)$ and $\mathcal{B} = (D_A, I_B)$ are said to elementarily equivalent if for all fo-sentences σ, $\mathcal{A} \models \sigma$ iff $\mathcal{B} \models \sigma$.

We write: $\mathcal{A} \equiv \mathcal{B}$.

Proposition: If two structures are isomorphic, then they are elementarily equivalent.

To talk about isomorphism between structures we have to discuss homomorphisms between structures, and of course, deal with injective and surjective functions. In the following we will first define homomorphisms, and then move on to show the relationship between isomorphism and elementary equivalence.

Homomorphism

Let $\mathcal{L} : (\mathcal{C}, \mathcal{F}, \mathcal{P})$ be a fo-language. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{L}$-structures. A homomorphism h of $\mathcal{A}$ into $\mathcal{B}$ is a function $h : D_{\mathcal{A}} \to D_{\mathcal{B}}$, s.t.

(i) $h(I_{\mathcal{A}}(c)) = I_{\mathcal{B}}(c)$ for all $c \in \mathcal{C}$.

(ii) $h(f^n_{\mathcal{A}}(a_1, \ldots, a_n)) = f^n_{\mathcal{B}}(h(a_1), \ldots, h(a_n))$, for all n-ary function symbols for all n.

(iii) $(a_1, \ldots, a_n) \in p^n_{\mathcal{A}}$ iff $(h(a_1), \ldots, h(a_n)) \in p^n_{\mathcal{B}}$ for all $a_1, \ldots, a_n \in D_{\mathcal{A}}$.

- If h is injective, h is an embedding.
- If h is surjective in addition, h is an isomorphism.

How would the terms behave under h?

Let y_A be an assignment of variables in $\mathcal{A}$. Then, for any term t, $h(y_A(t)) = y_B(t)$, where $y_B = h \circ y_A$.

H.W. Prove the statement above.

Theorem: Let $\mathcal{A}$, $\mathcal{B}$, $y_{\mathcal{A}}$ and $y_{\mathcal{B}}$ are given as above.

(i) For quantifier-free formulas φ, without equality, we have: $(\mathcal{A}, y_{\mathcal{A}}) \models \varphi$ iff $(\mathcal{B}, y_{\mathcal{B}}) \models \varphi$.

Proof: We prove this by induction on the size of φ.

Base case: $\varphi : p_i^n t_1 \cdots t_n$. $(\mathcal{A}, y_{\mathcal{A}}) \models \varphi$ iff $(\mathcal{A}, y_{\mathcal{A}}) \models p_i^n t_1 \cdots t_n$ iff $(y_{\mathcal{A}}(t_1), \ldots, y_{\mathcal{A}}(t_n)) \in p_{\mathcal{A}}^n$

iff $(h(y_{\mathcal{A}}(t_1)), \ldots, h(y_{\mathcal{A}}(t_n))) \in p_{\mathcal{B}}^n$ (as, h is a homomorphism)

iff $(y_{\mathcal{B}}(t_1), \ldots, y_{\mathcal{B}}(t_n)) \in p_{\mathcal{B}}^n$ iff $(\mathcal{B}, y_{\mathcal{B}}) \models p_i^n t_1 \cdots t_n$

iff $(\mathcal{B}, y_{\mathcal{B}}) \models \varphi$.

Induction Hypothesis: Suppose the result holds for all formulas of size $\leq n$.

Induction Step: Let the size of φ be $n+1$.

$\boxed{\text{Case I}}$: $\varphi := \neg\psi$. $(\mathcal{A}, y_{\mathcal{A}}) \models \varphi$ iff $(\mathcal{A}, y_{\mathcal{A}}) \models \neg\psi$

iff $(\mathcal{A}, y_{\mathcal{A}}) \not\models \psi$ iff $(\mathcal{B}, y_{\mathcal{B}}) \not\models \psi$ (by I.H.) iff $(\mathcal{B}, y_{\mathcal{B}}) \models \neg\psi$

iff $(\mathcal{B}, y_{\mathcal{B}}) \models \varphi$.

$\boxed{\text{Case II}}$: $\varphi := \psi \wedge \chi$. $(\mathcal{A}, y_{\mathcal{A}}) \models \varphi$ iff $(\mathcal{A}, y_{\mathcal{A}}) \models \psi \wedge \chi$

iff $(\mathcal{A}, y_{\mathcal{A}}) \models \psi$ and $(\mathcal{A}, y_{\mathcal{A}}) \models \chi$ iff $(\mathcal{B}, y_{\mathcal{B}}) \models \psi$ and $(\mathcal{B}, y_{\mathcal{B}}) \models \chi$ (by I.H.) iff $(\mathcal{B}, y_{\mathcal{B}}) \models \psi \wedge \chi$ iff $(\mathcal{B}, y_{\mathcal{B}}) \models \varphi$.

This completes the proof of (i).

(ii) Consider quantifier-free formulas with equality

[Assume h is injective]

Proof: It is enough to show the result for $\varphi := t_1 \equiv t_2$: $(\mathcal{A}, y_{\mathcal{A}}) \models t_1 \equiv t_2$ iff $y_{\mathcal{A}}(t_1) = y_{\mathcal{A}}(t_2)$ iff $h(y_{\mathcal{A}}(t_1)) = h(y_{\mathcal{A}}(t_2))$ iff $y_{\mathcal{B}}(t_1) = y_{\mathcal{B}}(t_2)$ iff $(\mathcal{B}, y_{\mathcal{B}}) \models t_1 \equiv t_2$.

(iii) Consider quantified formulas

[Assume h is surjective, in addition]

Proof: It is enough to show the result for $\varphi := \exists x \psi$. $(\mathcal{A}, y_{\mathcal{A}}) \models \exists x \psi \Rightarrow (\mathcal{A}, y_{\mathcal{A}[x \to d]}) \models \psi$ for some $d \in D_{\mathcal{A}} \Rightarrow (\mathcal{B}, y_{\mathcal{B}[x \to h(d)]}) \models \psi$ (by I.H.) $\Rightarrow$ $(\mathcal{B}, y_{\mathcal{B}}) \models \exists x \psi$. Conversely, $(\mathcal{B}, y_{\mathcal{B}}) \models \exists x \psi \Rightarrow$

$(\mathcal{B}, y_{\mathcal{B}[x \to d']}) \models \psi$ for some $d' \in D_{\mathcal{B}} \Rightarrow (\mathcal{B}, y_{\mathcal{B}[x \to h(d)]})$ $\models \psi$ for some $d \in D_{\mathcal{A}} \Rightarrow (\mathcal{A}, y_{\mathcal{A}[x \to d]}) \models \psi$ (by I.H.) $\Rightarrow (\mathcal{A}, y_{\mathcal{A}}) \models \exists x \psi$.

This completes the proof for all formulas.

What did we exactly prove here?

We proved the following result: Let $\mathcal{L}$ be an fo-language. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{L}$-structures. Let $h : D_{\mathcal{A}} \to D_{\mathcal{B}}$ be an isomorphism between the structures $\mathcal{A}$ and $\mathcal{B}$. Let $y_{\mathcal{A}}$ be an assignment function in the structure $\mathcal{A}$ and $y_{\mathcal{B}} = h \circ y_{\mathcal{A}}$ be the corresponding assignment function in $\mathcal{B}$. Then, for all fo-formulas φ, $(\mathcal{A}, y_{\mathcal{A}}) \models \varphi$ iff $(\mathcal{B}, y_{\mathcal{B}}) \models \varphi$.

Corollary: Isomorphic structures are elementarily equivalent.

What about the converse? Are elementarily equivalent structures isomorphic?

NO! (We will show that $(\mathbb{R}, <)$ and $(\mathbb{Q}, <)$ are elementarily equivalent. Of course they are not isomorphic)