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Case 3: $\varphi: \Diamond \psi$. Suppose $M, w \models \Diamond \psi$. Then, there exists v in M s.t. $w R v$ and $M, v \models \psi$. To show: $M_\sim, [w] \models \Diamond \psi$. We have to show that there is $[v]$ in $M_\sim$ s.t. $[w] R_\sim [v]$ and $M_\sim, [v] \models \psi$. Since, $M, v \models \psi$, by I.H., $M_\sim, [v] \models \psi$. So, if we can show that $[w] R_\sim [v]$, we are done. Let us assume the following condition on $R_\sim$:

(1) Condition of $R_\sim$: if $w R v$ then $[w] R_\sim [v]$.

Then we have $[w] R_\sim [v]$, and hence the required result, that is, $M_\sim, [w] \models \Diamond \psi$.

Conversely, suppose that $M_\sim, [w] \models \Diamond \psi$. To show: $M, w \models \Diamond \psi$. Since $M_\sim, [w] \models \Diamond \psi$, there is $[v]$ in $M_\sim$ s.t., $[w] R_\sim [v]$ and $M_\sim, [v] \models \psi$.

Once again, by I.H., $M, v \models \psi$. We need to show that $M, w \models \Diamond \psi$. We assume the following condition on $R_\sim$:

(2) Condition of $R_\sim$: If $[w] R_\sim [v]$, then for all $\Diamond \phi \in \text{Sub}(\varphi)$, if $M, v \models \phi$, then $M, w \models \Diamond \phi$.

This condition directly gives us $M, w \models \Diamond \psi$. This completes the proof once we get a binary relation $R_\sim$ on $W_\sim$ satisfying the conditions given above.

A definition of $R \sim$ satisfying conditions (1) & (2).

$[w] R \sim [v]$ iff there is $w' \in [w]$ and $v' \in [v]$, s.t.
 $w' R v'$.

H.W. (1) Check whether the conditions are satisfied.
(2) Come up with some other definition of $R \sim$ satisfying these conditions.

Modal logic satisfies SFMP

Why are we considering SFMP?

This gives decidability of modal logic.

Decidability of a logic.

A logic L is said to be decidable if the satisfiability problem of L is decidable.

What is a satisfiability problem?

Given a formula ϕ in L , is ϕ satisfiable?

Satisfiability problem for modal logic

Given a modal formula ϕ , does there exist a model $M: (W, R, V)$ and a world w in M , s.t. $M, w \models \phi$?

Is the SAT problem of modal logic decidable?

Yes, it is. Let ϕ be a formula whose satisfiability we would like to check. From SFMP, we get a bound $2^{|\phi|}$

s.t. if φ is satisfiable then φ is satisfiable in a model of size at most $2^{|\varphi|}$. We can consider all models of size $1, 2, \dots, 2^{|\varphi|}$ and check whether φ is satisfiable in any one of the models. We can construct a Turing machine to generate all such models of size at most $2^{|\varphi|}$. This gives us a finite algorithm to check whether φ is satisfiable. Thus, model logic is decidable.

Similar problems

Given a formula φ , is there a reflexive / symmetric / transitive model that satisfies φ ?

Are these problems decidable?

Let us first consider the reflexive models. Let φ be a formula. We want to check whether there is a reflexive model $M: (W, R, V)$ and a world w in M , s.t. $M, w \models \varphi$. How do we check? As earlier, we show that if φ is satisfiable in a reflexive model then φ is satisfiable in a reflexive model of bounded size. Let $M: (W, R, V)$ be a reflexive model satisfying φ . Then, we can construct $M_n: (W_n, R_n, V_n)$ as earlier.

Is R_n reflexive? Yes, it is. We show that $[w] R_n [w]$ for all $[w] \in W_n$, given $w R w$ for all $w \in W$. This follows from Condition (1).

H.W. Investigate the cases of symmetric and transitive models.

More model-theoretic properties.

Proposition: Let φ be a modal formula. If φ is satisfiable, then φ is satisfiable in a connected model.

Proof: Since φ is satisfiable, there is a model $M: (W, R, V)$ and a world w in M s.t. $M, w \models \varphi$.

We now construct a model M' as follows:

$$W' = \{w' \in W : w' \text{ is reachable from } w\}$$

$$R' = R \cap (W' \times W')$$

$$V'(p) = V(p) \cap W', \text{ for all propositional atom } p.$$

Evidently, M' is a connected model, that is, if $w' \in W'$ and $w' R w''$, then $w'' \in W'$.

Claim: For all modal formulas ψ and for all worlds w' in M' , $M, w' \models \psi$ iff $M', w' \models \psi$.

H.W. Prove this claim.

By the claim we have that $M', w \models \varphi$, as, $M, w \models \varphi$, hence the result. This completes the proof.

Tree-like model.

A model $M: (W, R, V)$ is tree-like if:

(i) there is a unique $w \in W$ s.t. $W = \{w' : w' \text{ is reachable from } w\}$

(ii) for all $w' \in W$ with $w' \neq w$, there is a unique w'' s.t.
 $w'' R w'$.

(ii) R is acyclic.

Proposition: Let φ be a modal logic formula. If φ is satisfiable, then φ is satisfiable in a tree-like model.

Proof: Since φ is satisfiable, there is a model $M: (W, R, V)$ and a world w in M s.t. $M, w \models \varphi$.

We now construct a tree-like model M' as follows:

$$W' = \{(w, u_1, \dots, u_n) : n \geq 0; w R u_1 R u_2 \dots R u_n\}$$

$$R' = \{((w, u_1, \dots, u_m), (w, v_1, \dots, v_l)) : l = m+1, u_i = v_i \text{ for all } i=1, \dots, m; u_m R v_l\}$$

$$V'(p) = \{(w, u_1, \dots, u_n) : u_n \in V(p)\} \text{ for all propositional atoms } p.$$

Claim: For all modal formulas ψ and for all worlds $(w, u_1, \dots, u_n)$ in M' , $M', (w, u_1, \dots, u_n) \models \psi$ iff $M, u_n \models \psi$.

H.W. Prove this claim.

By the claim we have that $M', w \models \varphi$, as, $M, w \models \varphi$, hence the result. This completes the proof.

We are ending our discussion on model-theoretic properties of modal logic and

move over to a discussion on **frames**.

From the perspectives of models $M: (W, R, V)$, we move on to the perspectives of frames $F: (W, R)$.

Given a frame F with a binary relation R , can we express properties of this relation R in terms of modal formulas?

Some notions of validity

* A modal formula ϕ is said to be valid if for every model M and every world w in M , $M, w \models \phi$.

* Given a model $M: (W, R, V)$, we call a formula ϕ **M -valid** ($M \models \phi$) if for all $w \in W$, $M, w \models \phi$.

* Given a frame $F: (W, R)$, we call a formula ϕ **F -valid** ($F \models \phi$) if for every model $M: (F, V)$, ϕ is M -valid.

* A modal formula ϕ is said to characterize a class of frames, $\mathcal{L}$, say, if $\mathcal{L} = \{F : \phi \text{ is } F\text{-valid}\}$.

Examples

1. Consider $\mathcal{L} = \{(W, R) : R \text{ is reflexive}\}$. Can we find a modal formula that characterizes $\mathcal{L}$?

Yes: $\Box p \rightarrow p$ (equivalently, $p \rightarrow \Diamond p$), p : propositional atom.

Claim: $\Box p \rightarrow p$ characterizes reflexive frames.

Proof: We need to show that for any frame F ,

$F \models \Box p \rightarrow p$ iff R_F is reflexive.

* Let F be frame s.t. R_F is reflexive. To show that:
 $F \models \Box p \rightarrow p$. Then, we need to show that for all models
 M based on F and for all worlds w in M , $M, w \models \Box p \rightarrow p$.
Take some M over F and some w in M . To show: $M, w \models \Box p \rightarrow p$.
Suppose $M, w \models \Box p$. To show $M, w \models p$. Since R_F
is reflexive, $w R_F w$. So, as $M, w \models \Box p$, $M, w \models p$.

* Conversely, suppose that F is a frame s.t. $F \not\models \Box p \rightarrow p$.
To show that R_F is not reflexive. We prove this contraposi-
tively. Suppose R_F is not reflexive. To show: $F \not\models \Box p \rightarrow p$.
It is enough to come up with a model M based
on the frame F and a world w in M , s.t. $M, w \not\models \Box p \rightarrow p$.
Now since R_F is not reflexive there is a world w'
in W s.t. $w' \not R_F w'$. Consider the model $M = (F, V)$,
where, $V(p) = W \setminus \{w'\}$. Then, $M, w' \models \Box p$, but, M ,
 $w' \not\models p$. So, $M, w' \not\models \Box p \rightarrow p$. So, $F \not\models \Box p \rightarrow p$. This completes
the proof of our claim.

2. Consider $\mathcal{A} = \{(W, R) : R \text{ is transitive}\}$ Can we find
a modal formula that characterizes $\mathcal{A}$?

Yes. $\Box p \rightarrow \Box \Box p$ (equivalently, $\Diamond \Diamond p \rightarrow \Diamond p$), p : prop. atom.

Claim: $\Box p \rightarrow \Box \Box p$ characterizes transitive frames.

Proof: We need to show that for any frame F , $F \models \Box p \rightarrow \Box \Box p$ iff R_F is transitive.

* Suppose F be a frame, s.t. R_F is transitive. To show that $F \models \Box p \rightarrow \Box \Box p$. Let M be a model based on F and w be a world in M . To show: $M, w \models \Box p \rightarrow \Box \Box p$. Let $M, w \models \Box p$. We have to show $M, w \models \Box \Box p$. Now, $M, w \models \Box \Box p$ if for all v in M with $w R_F v$, $M, v \models \Box p$, and $M, v \models \Box p$ if for all u in M with $v R_F u$, $M, u \models p$ ———— (*). Since, R_F is transitive, whenever $w R_F v$ and $v R_F u$, we have: $w R_F u$. Then, since $M, w \models \Box p$, $M, u \models p$. To show (*): take any v s.t. $w R_F v$ and u s.t. $v R_F u$, we have by our assumption above, $M, u \models p$. So, we have, $M, w \models \Box \Box p$, and so, $M, w \models \Box p \rightarrow \Box \Box p$. Thus, $F \models \Box p \rightarrow \Box \Box p$.

* Conversely, suppose that $F \models \Box p \rightarrow \Box \Box p$. To show that R_F is transitive. Suppose not. We show: $F \not\models \Box p \rightarrow \Box \Box p$. It is enough to come up with a model $M: (F, V)$ and a world w in M , s.t. $M, w \not\models \Box p \rightarrow \Box \Box p$. Since R_F is not transitive, there are worlds $w', v', u' \in W$, s.t. $w' R_F v'$ and $v' R_F u'$ but $w' \not R_F u'$. Now, consider a valuation V , s.t. $V(p) = W \setminus \{u'\}$. Then, we have $M, w' \models \Box p$ and $M, w' \not\models \Box \Box p$. So, $M, w' \not\models \Box p \rightarrow \Box \Box p$.

Hence $F \models \Box p \rightarrow \Box \Box p$. This completes the proof of the claim.

H.W.

1. Characterize the class of all symmetric frames.
2. Characterize the class of all serial frames.
3. What conditions on R_F do the following formulas characterize? Justify your answer.

(a) $\Diamond p \rightarrow \Box p$

(b) $\Diamond p \rightarrow \Box \Diamond p$

(c) $\Box \Box p \rightarrow \Box p$

(d) $\Box (\Box p \rightarrow p)$

(e) $\Diamond \Box p \rightarrow \Box \Diamond p$

We end the discussion today by focussing on the following question.

What is the main difference between considering usual model validities and the frame validities or the frame characterisations today?

The two examples we worked on today are first-order definable (Check!)

Are such relational properties always fo-definable?