

15.10.2026

Let us consider one more example. Let  $F = (W, R)$  be a frame where  $R$  is *transitive* and *conversely well-founded*. Before going any further, let us explain what do we mean by converse well-foundedness.

A relation  $R$  on a set  $W$  is said to be *well-founded* if there is no infinite sequence of the form  $\dots w_3 R w_2 R w_1 R w_0$ .  $R$  is *conversely well-founded* if there is no infinite sequence of the form  $w_0 R w_1 R w_2 R w_3 R \dots$ .

*Claim* :  $\Box(\Box p \rightarrow p) \rightarrow \Box p$  characterizes *transitive and conversely well-founded frames*.

*Proof* : We need to show that for any frame  $F$ ,  $F \models \Box(\Box p \rightarrow p) \rightarrow \Box p$  iff  $R_F$  is transitive and conversely well-founded.

\* Suppose  $R_F$  is transitive and conversely well-founded. To show:  $F \models \Box(\Box p \rightarrow p) \rightarrow \Box p$ . Suppose not. Then, there is a valuation  $V$  and a world  $w$  in  $W_F$  such that  $(F, V), w \not\models \Box(\Box p \rightarrow p) \rightarrow \Box p$ . Let us denote  $(F, V)$  by  $M$ .

So, we have :  $M, w \models \Box(\Box p \rightarrow p)$ , but,  $M, w \not\models \Box p$ .  
 Then, there is  $w_1 \in W_F$  s.t.  $w R w_1$  and  $M, w_1 \not\models p$ .  
 Then,  $M, w_1 \not\models \Box p$  as,  $M, w_1 \models \Box p \rightarrow p$  (since,  $M, w \models \Box(\Box p \rightarrow p)$ ). Then, there is  $w_2 \in W_F$  s.t.  $w_1 R w_2$  and  $M, w_2 \not\models p$ . Note that, as  $R_F$  is transitive,  $w R w_2$ . Repeating the argument we have  $w_3$  s.t.  $w_2 R w_3$  and  $M, w_3 \not\models p$ . Continuing like this we have an infinite path  $w R_F w_1, R_F w_2 R_F w_3 R_F \dots$ , contradicting the converse well-foundedness of  $R_F$ . So,  $F \models \Box(\Box p \rightarrow p) \rightarrow \Box p$ .

★ Conversely, suppose that  $F \models \Box(\Box p \rightarrow p) \rightarrow \Box p$ . To show that  $R_F$  is transitive and conversely well-founded.

We prove this by contraposition. We have two cases.

Case 1. Suppose  $R_F$  is not transitive. We show

that  $F \not\models \Box(\Box p \rightarrow p) \rightarrow \Box p$ , in fact, we show :

$F \not\models \Diamond p \rightarrow \Diamond(p \wedge \neg \Diamond p)$  (equivalent to the formula

$\Box(\Box p \rightarrow p) \rightarrow \Box p$ ) **H.W.** Complete the proof.

Case 2. Suppose  $R_F$  is not conversely well-founded.

We show that  $F \not\models \Box(\Box p \rightarrow p) \rightarrow \Box p$ . It is enough

to come up with a model  $M: (F, V)$  and a world

$w$  in  $M$  s.t.  $M, w \not\models \Box(\Box p \rightarrow p) \rightarrow \Box p$ . Define  $V(p)$

$= W_F \setminus \{w \in W_F : \text{there is an infinite sequence starting from } w.\}$

Let  $w' \in W_F$  s.t. there is an infinite sequence starting from  $w'$ , that is,  $w' R w'' R w''' R \dots$

Then,  $M, w' \not\models \Box p$ . Also, for any world  $w \in W_F$ ,

$M, w \models \Box p \rightarrow p$  [Take any  $w$ . Suppose  $M, w \models \Box p$ .

Then, for all  $w'$  with  $w R w'$ ,  $M, w' \models p$ . Thus, there cannot be any infinite sequence starting from  $w$ .

Hence,  $M, w \models p$ . Thus,  $M, w \models \Box p \rightarrow p$ ]. So,

$M, w' \models \Box(\Box p \rightarrow p)$ . But  $M, w' \not\models \Box p$ . So,  $M, w' \not\models \Box(\Box p \rightarrow p) \rightarrow \Box p$ .

This completes the proof.

Why we are considering this example?

The properties like reflexivity and transitivity are first-order definable. However, that is not the case here.

**Proposition.** The class of frames defined by the formula  $\Box(\Box p \rightarrow p) \rightarrow \Box p$  is not fo-definable.

**Proof:** Let  $\alpha := \Box(\Box p \rightarrow p) \rightarrow \Box p$ . Suppose there

is a first-order formula  $\varphi$  equivalent to  $\alpha$ .

Then, any model where  $\varphi$  holds has to be

**transitive**. Let  $\varphi_n(x_0, \dots, x_n)$  be the fo-formula

stating that there is an  $R$ -path of length  $n$

through  $x_0, \dots, x_n : \varphi_n(x_0, \dots, x_n) = \bigwedge_{0 \leq i < n} R x_i x_{i+1}$

Let  $\Sigma = \{\varphi\} \cup \{\forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz)\} \cup \{\varphi_n : n \in \omega\}$

Then, every finite subset of  $\Sigma$  is satisfiable.

So,  $\Sigma$  is satisfiable. But, a model of  $\Sigma$

cannot be conversely well-founded. However,  $\varphi$ , which is equivalent to  $\varphi$ , defines the class of transitive conversely well-founded frames. This contradiction shows us that  $\varphi$  cannot be equivalent to a first-order formula. This completes the proof.

What is happening here?

\* With respect to models, modal logic sits inside first-order logic.

\* With respect to frames, modal logic goes beyond first-order logic.

Remark: There are many questions here which need to be studied in details.

We now get into reasoning in modal logic.

## Consequence relations

Let  $\Gamma$  be a set of modal formulae and  $\phi$  be a formula.

### Semantic Consequence Relation ( $\Gamma \models \phi$ )

For all Kripke models  $M$  and for all worlds  $w$  in  $M$ , if  $M, w \models \gamma$  for all  $\gamma \in \Gamma$ , then  $M, w \models \phi$ .

Note: When  $\Gamma = \emptyset$ ,  $\phi$  is said to be **valid**, and denoted by  $\models \phi$ .

### Deductive Consequence Relation ( $\Gamma \vdash \phi$ )

If there is a sequence of formulae  $\phi_1, \phi_2, \dots, \phi_n$ , such that: (i)  $\phi_n = \phi$ , (ii) each  $\phi_i$  is either an axiom or, a member of  $\Gamma$  or, obtained by application of some rule.

Note: When  $\Gamma = \emptyset$ ,  $\phi$  is said to be **theorem**, and denoted by  $\vdash \phi$ .

Main goal: Prove soundness and completeness, that is prove:  $\Gamma \vdash \phi$  iff  $\Gamma \models \phi$ .

**H.W.** Prove the soundness theorem for modal logic.

## Completeness Theorem

Suppose  $\Gamma \not\models \phi$ . To show that  $\Gamma \vdash \phi$ . Suppose not, that is,  $\Gamma \not\vdash \phi$ . We introduce the notion of consistent sets of formulas as earlier.

**Consistent Set**: A set of formulas  $\Sigma$  is said to be **inconsistent** if there is a formula  $\phi$  such that  $\Sigma \vdash \phi$  and  $\Sigma \vdash \neg \phi$ .  $\Sigma$  is **consistent** if it is not inconsistent.

Given the notion of consistency, we have that, since  $\Gamma \not\vdash \phi$ ,  $\Gamma \cup \{\neg \phi\}$  is consistent. (check!)

**Claim**: Any consistent set of formulas has a model.

Using this claim, we have that  $\Gamma \cup \{\neg \phi\}$  has a model contradicting the fact that  $\Gamma \not\models \phi$ . So,  $\Gamma \vdash \phi$ .

This completes the proof.

## Proof of the claim

Let  $\Gamma$  be a consistent set of modal formulas. As earlier, we do the following:

**Step 1**: Extend  $\Gamma$  to a consistent and complete set (maximally consistent set)  $\Delta$ , say. (H.W.)

The step 1 above is what we call **Lindenbaum Lemma**.

We first note that the maximally consistent (mc) set of formulas should have the natural properties of the Boolean connectives as mentioned earlier: Let  $\Gamma$  be an mc set.

1. (a) For every formula  $\phi$ ,  $\phi \in \Gamma$  or  $\neg\phi \in \Gamma$ .

(b) Both  $\phi$  and  $\neg\phi$  cannot belong to  $\Gamma$ .

2. (a)  $\phi \wedge \psi \in \Gamma$  iff ( $\phi \in \Gamma$  and  $\psi \in \Gamma$ )

(b)  $\phi \vee \psi \in \Gamma$  iff ( $\phi \in \Gamma$  or  $\psi \in \Gamma$ )

(c)  $\phi \rightarrow \psi \in \Gamma$  iff ( $\phi \in \Gamma$  implies  $\psi \in \Gamma$ )

(d)  $\phi \leftrightarrow \psi \in \Gamma$  iff ( $\phi \in \Gamma$  iff  $\psi \in \Gamma$ )

To have all these properties for the mc's, we would need the **Classical Propositional**

**Logic** axiom system. To include them in the axiom system **modal logic**, we consider:

### Substitution

A substitution is a map  $\sigma : \mathcal{P} \rightarrow \mathcal{L}$ , where  $\mathcal{P}$  is the set of propositional variables and  $\mathcal{L}$  is the set of modal formulas. Given  $\sigma$ , we have  $\hat{\sigma} : \mathcal{L} \rightarrow \mathcal{L}$ :

$$\hat{\sigma}(p) = \sigma(p); \quad \hat{\sigma}(\perp) = \perp \quad ; \quad \hat{\sigma}(\neg \phi) = \neg \hat{\sigma}(\phi);$$

$$\hat{\sigma}(\phi \wedge \psi) = \hat{\sigma}(\phi) \wedge \hat{\sigma}(\psi) \quad ; \quad \hat{\sigma}(\phi \vee \psi) = \hat{\sigma}(\phi) \vee \hat{\sigma}(\psi);$$

$$\hat{\sigma}(\phi \rightarrow \psi) = \hat{\sigma}(\phi) \rightarrow \hat{\sigma}(\psi) \quad ; \quad \hat{\sigma}(\phi \leftrightarrow \psi) = \hat{\sigma}(\phi) \leftrightarrow \hat{\sigma}(\psi);$$

$$\hat{\sigma}(\Box \phi) = \Box \hat{\sigma}(\phi) \quad ; \quad \hat{\sigma}(\Diamond \phi) = \Diamond \hat{\sigma}(\phi).$$

**Remark:**  $\sigma$  can be extended uniquely to  $\hat{\sigma}$ , so we denote  $\hat{\sigma}$  by  $\sigma$ .

**Examples:**  $\sigma(p) = \Box \Diamond (p \wedge q); \quad \sigma(q) = p \wedge \Box q$

**H.W.** Find: (i)  $\sigma(\Box(p \wedge q) \rightarrow \Box p)$   
(ii)  $\sigma(p \rightarrow (q \rightarrow p))$

**Propositional tautology in modal logic.**

A modal formula  $\phi$  is a propositional tautology if  $\phi = \sigma(\alpha)$ , where  $\alpha$  is a tautology in CPL and  $\sigma$  is a substitution function.

$$p \rightarrow (q \rightarrow p) \quad \Box p \rightarrow (q \rightarrow \Box p)$$

Thus, we are ready with our first set of axioms and rules for the axiom system of modal logic:

## Axioms

(1) All propositional tautologies

## Rules

(1) Modus Ponens

**Step 2** Show that  $\Delta$  has a model

Thus, we now focus on finding a model for the mcs  $\Delta \supseteq \Gamma$  which would show us that  $\Gamma$  has a model and the proof be complete. It is enough to find a model  $M = (W, R, V)$  and a world  $w \in W$  such that the following holds:

$$M, w \models \varphi \text{ iff } \varphi \in \Delta$$

**Truth Lemma.**

How do we construct such an  $M$ ?

We need to define  $M$ , and consider a  $w$  in  $M$  such that the truth lemma holds.

Also, the resulting pointed model should also satisfy the conditions of  $M, w \models \varphi$  for all modal formulas  $\varphi$ . We leave it for the next class.