$\boxed{28.07.2026}$

Proposition: Let $\mathcal{L}$ be an f.o.-language, and $\mathcal{A}$ be an $\mathcal{L}$-structure. Let R be an n-ary relation on $D_{\mathcal{A}}$ such that R is definable in $\mathcal{L}$. Let h be an automorphism on $\mathcal{A}$. Then, $(a_1, \ldots, a_n) \in R$ iff $(h(a_1), \ldots, h(a_n)) \in R$.

Proof: Let $\varphi(x_1, \ldots, x_n)$ be a formula that defines R in $\mathcal{A}$. Then we have: $(a_1, \ldots, a_n) \in R$

iff $\mathcal{A}_{[x_1 \to a_1, \ldots, x_n \to a_n]} \models \varphi$.

iff $\mathcal{A}_{[x_1 \to h(a_1), \ldots, x_n \to h(a_n)]} \models \varphi$ (by the result above)

iff $(h(a_1), \ldots, h(a_n)) \in R$. This completes the proof.

H.W. Using this result show that $\{b\}$ is not definable in the example on graphs given earlier.

Another example

Consider the structure $(\mathbb{R}, <)$. Of course, $\mathbb{N} \subseteq \mathbb{R}$. We show that $\mathbb{N}$ is not definable in $\mathbb{R}$, given the language $\mathcal{L}$ having a binary relation symbol p, say whose interpretation in $\mathbb{R}$ is given by $<$. We can show this by using an automorphism on $\mathbb{R}$: $h(x) = x^3$. Since h is automorphism, had $\mathbb{N}$ been definable, we should have: $x \in \mathbb{N}$ iff $h(x) \in \mathbb{N}$. But, there are elements outside $\mathbb{N}$, which get mapped in $\mathbb{N}$. Thus, $\mathbb{N}$ is not definable in $(\mathbb{R}, <)$.

Till now, we have focussed on fo-language and expressivity. We will come back to these concepts, but for now let us dive into the other important aspect of any logic, that is 'reasoning' in a given language..

(Semantic) Consequence relation

Γ : set of formulas

φ : a formula.

We say that φ is a semantic consequence of Γ ($\Gamma \models \varphi$) if for all models $\mathcal{M}$, $\mathcal{M} \models \gamma$ for all $\gamma \in \Gamma$ imply $\mathcal{M} \models \varphi$.

$\mathcal{M} \models \Gamma$

Example

$\Gamma = \{ px \to qx , qx \to rx \}$

$\varphi := px \to rx$.

To show : $\Gamma \models \varphi$.

We need to show that : for all models $\mathcal{M}$, if $\mathcal{M} \models \Gamma$, then $\mathcal{M} \models \varphi$.

Take any model $\mathcal{M} : (D, I, g)$. Suppose $\mathcal{M} \models \Gamma$, that is, $\mathcal{M} \models px \to qx$ and $\mathcal{M} \models qx \to rx$

To show : $\mathcal{M} \models px \to rx$.

Suppose $\mathcal{M} \models px$. Then, to show $\mathcal{M} \models rx$.

Since $\mathcal{M} \models px$, $\mathcal{M} \models qx$ (as, $\mathcal{M} \models px \to qx$)

Then, $\mathcal{M} \models rx$ (as, $\mathcal{M} \models qx \to rx$).

This completes the proof.

Now, suppose $\Gamma = \Phi$. Then, $\Gamma \models \varphi$ means $\Phi \models \varphi$ (we write $\models \varphi$), which basically says, 'φ is satisfied by all models'.

- A formula φ is said to be valid if for all models $\mathcal{M}$, $\mathcal{M} \models \varphi$.

- A formula φ is said to be satisfiable if there is a model $\mathcal{M}$, such that, $\mathcal{M} \models \varphi$.

Example on truth, validity, satisfiability.

Consider an fo-language $\mathcal{L}$ with $\mathcal{C} = \phi$, $\mathcal{F} = \phi$, $\mathcal{P} = \{p_1^2\}$. Consider the structures (D, I_k), where, D is a non-empty set and I_k for $k = 1, 2, 3$ be given by :

$I_2(p_1^2) = \phi$, $I_1(p_1^2) = D \times D$.

$I_1(p_1^2) = D \times D \xleftrightarrow[\text{formula}]{\text{true}} \varphi_1 := \forall x \forall y\, p_1^2 xy$

$I_2(p_1^2) = \phi \xleftrightarrow[\text{formula}]{\text{true}} \varphi_2 := \forall x \forall y\, \neg p_1^2 xy$

$I_3(p_1^2) =$ transitive relation $\xleftrightarrow[\text{formula}]{\text{true}} \varphi_3 := \forall x \forall y \forall z\, ((p_1^2 xy \wedge p_1^2 yz) \to p_1^2 xz)$

Valid and satisfiable formulas

- $\forall x \forall y\, (p_1^2 xy \vee \neg p_1^2 xy)$: valid
- $\forall x \forall y\, (p_1^2 xy \wedge \neg p_1^2 xy)$: satisfiable only in empty domain.
- $\exists x \exists y\, (p_1^2 xy \wedge \neg p_1^2 xy)$: unsatisfiable.

H.W.

- Let $\Gamma = \{\varphi, \neg\varphi\}$. Show that $\Gamma \models \psi$ for all formulas ψ.

- Show the following:

(a) If $\varphi \in \Gamma$, then $\Gamma \models \varphi$

(b) φ is valid iff $\neg\varphi$ is not satisfiable.

(c) If $\Gamma_1 \models \varphi$ and $\Gamma_1 \subseteq \Gamma_2$, then $\Gamma_2 \models \varphi$.

(d) If $\Gamma \models \Delta$, and $\Delta \models \varphi$, then $\Gamma \models \varphi$

- $\Gamma \cup \{\varphi\} \models \psi$ iff $\Gamma \models \varphi \to \psi$: Prove or disprove.

- Check whether the following formulas are valid.

(a) $\forall x\,(px \to (qx \to px))$

(b) $\exists x\, px \wedge \exists x\, qx \to \exists x\,(px \wedge qx)$

(c) $\forall x\,(px \vee qx) \to \forall x\, px \vee \forall x\, qx$.

(d) $\exists x\,(px \wedge qx) \to \exists x\, px \wedge \exists x\, qx$.

(e) $\forall x\, px \vee \forall y\, py \to \forall x\,(px \vee qx)$

(f) $\forall x\,(px \to qx) \to (\forall x\, px \to \forall x\, qx)$

(g) $\exists x\,(px \to qx) \to (\exists x\, px \to \exists x\, qx)$

Remark: We now have some idea regarding the interaction of syntax and semantics. Before moving any further, let us warn ourselves about the usage of the symbol $\models$ that we have used in describing satisfaction as well as entailment.

$\mathcal{M} \models \varphi$ (satisfies)

$\Gamma \models \varphi$. (entails)

Models and Theories

- Given any formula φ, define $\mathrm{Mod}(\varphi) = \{\mathcal{M} : \mathcal{M} \models \varphi\}$. Given a set of formulas Γ, define $\mathrm{Mod}(\Gamma) = \{\mathcal{M} : \mathcal{M} \models \Gamma\}$.
- Let $\mathcal{K}$ be a class of models. The theory of $\mathcal{K}$, $\mathrm{Th}(\mathcal{K}) = \{\varphi : \mathcal{M} \models \varphi \text{ for all } \mathcal{M} \text{ in } \mathcal{K}\}$.

1. Let Γ_1 and Γ_2 be two sets of formulas s.t. $\Gamma_1 \subseteq \Gamma_2$. Then, $\mathrm{Mod}(\Gamma_1) \supseteq \mathrm{Mod}(\Gamma_2)$.

2. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two classes of models s.t. $\mathcal{K}_1 \subseteq \mathcal{K}_2$. Then, $\mathrm{Th}(\mathcal{K}_1) \supseteq \mathrm{Th}(\mathcal{K}_2)$

Consequence of a set of formulas.

Let Γ be a set of formulas. Then, consequence of Γ, $Con(\Gamma) = Th(\mathcal{M}od(\Gamma))$.

Proposition: Let Γ be a set of formulas and φ be a formula. Then, $\varphi \in Con(\Gamma)$ iff $\Gamma \models \varphi$.

H.W. Prove the proposition.

H.W. Prove the following properties of $Con(\Gamma)$.

1. $\Gamma \subseteq Con(\Gamma)$
2. if $\Gamma_1 \subseteq \Gamma_2$, then $Con(\Gamma_1) \subseteq Con(\Gamma_2)$
3. $Con(Con(\Gamma)) = Con(\Gamma)$.

What does it mean to say that $\Gamma \not\models \varphi$?

$\Gamma \not\models \varphi$ iff there is a model $\mathcal{M}$, s.t. $\mathcal{M} \models \Gamma$ but $\mathcal{M} \not\models \varphi$.

iff there is a model $\mathcal{M}$, s.t. $\mathcal{M} \models \Gamma$ and $\mathcal{M} \models \neg\varphi$

iff there is a model $\mathcal{M}$, s.t. $\mathcal{M} \models \Gamma \cup \{\neg\varphi\}$

iff $\Gamma \cup \{\neg\varphi\}$ is satisfiable.

More on satisfiability

Let Γ be a set of formulas, φ be a formula.

- φ is sat if there is a model of φ.
- Γ is sat if there is a model of Γ.

Example

Consider an fo-language $\mathcal{L}$ with unary relation symbols $p_1^1, p_2^1, p_3^1, \ldots\ldots$.

1. $\Gamma = \{p_1^1 x\}$. Is Γ sat?

We have to find a model (D, I, g) s.t. $(D, I, g) \models p_1^1 x$

$D = \{a, b\}$. $I(p_1^1) = \{a\}$, $g : V \to D : g(y) = a$ for all y.

Then, $(D, I, g) \models p_1^1 x$ as $g(x) \in I(p_1^1)$.

Thus, Γ is satisfiable.

2. $\Gamma = \{p_1^1 x, \neg p_1^1 x\}$. Is Γ sat?

Γ is unsatisfiable.

3. $\Gamma = \{p_1^1 x, p_2^1 x, \neg(p_1^1 x \wedge p_2^1 x)\}$. Is Γ sat?

Γ is unsatisfiable.

4. $\Gamma = \{p_1^1 x, p_2^1 x, p_3^1 x, \neg(p_1^1 x \wedge p_2^1 x \wedge p_3^1 x)\}$. Is Γ sat?

Γ is unsatisfiable.

Similarly, we can find unsatisfiable sets of formulas of any finite size k with $k \geqslant 2$.

What about infinite sets of formulas?

In the same way, if a set has any of

the above finite collections of formulas.

What about an infinite set Γ of formulas s.t. all finite subsets of Γ are sat?

Compactness Theorem of F.O.L

Let Γ be an infinite set of formulas. Then, Γ is sat iff every finite subset of Γ is sat (Γ is fin-sat).

Non-trivial part: If Γ is fin-sat, then Γ is sat.

How does this result connect with the consequence relation?

① If Γ is fin-sat then Γ is sat.

② If $\Gamma \models \varphi$, then there is a finite subset Γ_{fin} of Γ s.t. $\Gamma_{fin} \models \varphi$.

Proposition: ① iff ②

Proof: ② $\Rightarrow$ ①: Let Γ be fin-sat. To show that Γ is sat. Suppose not. Then, $\Gamma \models \varphi$ for all formulas φ. Then there is a formula ψ, say, s.t. $\Gamma \models \psi$ and $\Gamma \models \neg\psi$. Then, there are $\Gamma_1, \Gamma_2 \subseteq_{fin} \Gamma$ s.t. $\Gamma_1 \models \psi$ and $\Gamma_2 \models \neg\psi$. (by ②). So,

$\Gamma_1 \cup \Gamma_2 \models \psi \wedge \neg \psi$ and $\Gamma_1 \cup \Gamma_2 \subseteq_{fin} \Gamma$. Thus, $\Gamma_1 \cup \Gamma_2$ is unsatisfiable. This contradicts the assumption that Γ is fin-sat. This completes the proof.

(1) $\Rightarrow$ (2) : Let $\Gamma \models \varphi$. To show that there is $\Gamma_f \subseteq_{fin} \Gamma$ s.t. $\Gamma_f \models \varphi$. Suppose not. So, for all $\Gamma' \subseteq_{fin} \Gamma$, $\Gamma' \not\models \varphi$. So, for all $\Gamma' \subseteq_{fin} \Gamma$, $\Gamma' \cup \{\neg \varphi\}$ is sat. So, by (i) $\Gamma \cup \{\neg\varphi\}$ is sat. Then, $\Gamma \not\models \varphi$, a contradiction. This completes the proof

We will now get into providing a proof of compactness theorem in the next page.

Non-trivial part of Compactness Theorem

- Let Γ be an infinite set of formulas. If Γ is fin-sat then Γ is sat.

Proof : Every finite subset of Γ is satisfiable. To show : Γ is satisfiable
To show Γ is sat, we have to show that Γ has a model. So, we need to construct a model $(D, I, \mathcal{G})$ s.t.
$(D, I, \mathcal{G}) \models \Gamma$ $((D, I, \mathcal{G}) \models \gamma$ for all $\gamma \in \Gamma)$

Let us first ask the following question:
What properties should Γ have for it to have a model? To answer this question, let us now look at a set of formulas which is satisfied by a model.
Let $\mathcal{M}$ be a model and let $\Delta_{\mathcal{M}} = \{\varphi \mid \mathcal{M} \models \varphi\}$
What properties do $\Delta_{\mathcal{M}}$ have?

1. For any formula φ,
(a) $\Delta_{\mathcal{M}}$ contains φ or $\neg\varphi$.
(b) $\Delta_{\mathcal{M}}$ does not contain both.

(contd.)

2. For any formulas φ and ψ,

(a) $\varphi \vee \psi \in \Delta_M$ iff $\varphi \in \Delta_M$ or $\psi \in \Delta_M$

(b) $\varphi \wedge \psi \in \Delta_M$ iff $\varphi \in \Delta_M$ and $\psi \in \Delta_M$

(c) $\varphi \rightarrow \psi \in \Delta_M$ iff $\psi \in \Delta_M$ whenever $\varphi \in \Delta_M$

(d) $\varphi \leftrightarrow \psi \in \Delta_M$ iff $\varphi \in \Delta_M$ iff $\psi \in \Delta_M$

- A set of formulas satisfying conditions 1 and 2 is said to be a model set.

- A set of formulas Δ is complete if for any formula φ, $\varphi \in \Delta$ or $\neg\varphi \in \Delta$.

We will try to get a model of the fin-sat set Γ that we started with by the following 3 claims:

Claim 1: Every fin-sat set of formulas can be extended to a fin-sat and complete set of formulas.

Claim 2: Every fin-sat and complete set of formulas is a model set

Claim 3: Every model set is satisfiable