$\boxed{30.07.2026}$

$$\boxed{\underset{\text{fin-sat}}{\Gamma} \Longrightarrow \underset{\substack{\text{fin-sat} \\ + \\ \text{complete}}}{\Delta} \Longrightarrow \text{sat}}$$

Since $\Gamma \subseteq \Delta$, if Δ has a model then Γ will have a model, too. So, Γ becomes sat

Proof of claim 1:

We have that our language (set of all formulas) is countable $\boxed{\text{PROVE THIS!}}$

Then we can enumerate the set of formulas as follows:

$\varphi_0, \varphi_1, \varphi_2, \varphi_3, \ldots\ldots\ldots$

Let Γ be a fin-sat set of formulas.
Now, we construct a sequence of sets of formulas as follows:

$$\Gamma_0 = \Gamma$$

$$\Gamma_{k+1} = \begin{cases} \Gamma_k \cup \{\varphi_k\} & \text{if } \Gamma_k \cup \{\varphi_k\} \text{ is fin-sat} \\ \Gamma_k, & \text{otherwise} \end{cases}$$

So, $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \ldots\ldots$

Let $\Delta = \bigcup_{n \geq 0} \Gamma_n$

We claim that Δ is fin-sat and complete

- Let us first show that Δ is complete.

It is enough to show that for any set of formulas Γ' and any formula φ, if Γ' is fin-sat then either $\Gamma' \cup \{\varphi\}$ is fin-sat or $\Gamma' \cup \{\neg\varphi\}$ is fin-sat.

Proof: We have: Γ' is fin-sat. Suppose neither $\Gamma' \cup \{\varphi\}$ nor $\Gamma' \cup \{\neg\varphi\}$ is fin-sat.

So, there exist $\Gamma_1 \subseteq_{fin} \Gamma' \cup \{\varphi\}$, $\Gamma_2 \subseteq_{fin} \Gamma' \cup \{\neg\varphi\}$, s.t. Γ_1 and Γ_2 are not satisfiable. Then,

$\Gamma_1 = \Gamma_1' \cup \{\varphi\}$, $\Gamma_1' \subseteq_{fin} \Gamma'$

$\Gamma_2 = \Gamma_2' \cup \{\neg\varphi\}$, $\Gamma_2' \subseteq_{fin} \Gamma'$

Now, since $\Gamma_1' \cup \{\varphi\}$ is not sat,

$\Gamma_1' \models \neg\varphi$.

Also, since $\Gamma_2' \cup \{\neg\varphi\}$ is not sat,

$\Gamma_2' \models \varphi$.

So, $\Gamma_1' \cup \Gamma_2' \models \varphi \wedge \neg\varphi$.

Then, $\Gamma_1' \cup \Gamma_2'$ is not sat, a contradiction as $\Gamma_1' \cup \Gamma_2' \subseteq_{fin} \Gamma'$. So, either $\Gamma' \cup \{\varphi\}$ is sat. or $\Gamma' \cup \{\neg\varphi\}$ is sat.

- Δ is fin-sat

  Suppose Δ is not fin-sat. Then, there is $\Delta' \subseteq_{fin} \Delta$ s.t. Δ' is not sat.

  Now, $\Delta' \subseteq_{fin} \Gamma_k$ for some $k \geqslant 0$.

  Then, Γ_k cannot be fin-sat, a contradiction to the construction of Γ_k

  Hence, Δ is fin-sat.

This completes the proof of Claim 1.

Remark: This is a version on Lindenbaum Lemma: Any consistent set can be extended to a consistent and complete set (maximal consistent set). The notion of 'consistency' will be discussed later.

Proof of Claim 2:

Let Δ be a fin-sat and complete set of formulas. To show that Δ is a model set. Now, since Δ is fin-sat and complete, conditions 1.(a) and 1(b) are satisfied. We now show conditions 2(a) - 2(d).

Condition 2(a): To show that, $\varphi \vee \psi \in \Delta$ iff $\varphi \in \Delta$ or $\psi \in \Delta$.

- Suppose $\varphi \vee \psi \in \Delta$.
  To show $\varphi \in \Delta$ or $\psi \in \Delta$.
  Suppose not. Then $\varphi \notin \Delta$ and $\psi \notin \Delta$
  So, $\neg\varphi \in \Delta$ and $\neg\psi \in \Delta$.
  Then, $\{\neg\varphi, \neg\psi, \varphi \vee \psi\} \subseteq_{fin} \Delta$
  But, $\{\neg\varphi, \neg\psi, \varphi \vee \psi\}$ is unsatisfiable. This contradicts the fact that Δ is fin-sat. Hence, $\varphi \in \Delta$ or $\psi \in \Delta$.
- Conversely, suppose $\varphi \in \Delta$ or $\psi \in \Delta$
  W.l.o.g, let $\varphi \in \Delta$.
  To show $\varphi \vee \psi \in \Delta$

Suppose not. Then, $\neg(\varphi \vee \psi) \in \Delta$

Then $\{\varphi, \neg(\varphi \vee \psi)\} \subseteq_{fin} \Delta$.

But, $\{\varphi, \neg(\varphi \vee \psi)\}$ is unsatisfiable. This contradicts that Δ is fin-sat. Hence $\varphi \vee \psi \in \Delta$.

This completes the proof of 2(a).

H.W. Prove conditions 2(b) – 2(d).

This completes the proof of Claim 2.

Proof of Claim 3.

Let Δ be a model set. To show that Δ is sat, that is, to find a model $\mathcal{M}_\Delta$ s.t. $\mathcal{M}_\Delta \models \Delta$. Thus we need to find $\mathcal{M}_\Delta = (D_\Delta, I_\Delta, g_\Delta)$ s.t. :

$$\boxed{\mathcal{M}_\Delta \models \varphi \text{ iff } \varphi \in \Delta}$$

Suppose we have $\mathcal{M}_\Delta$, somehow.

To prove $\mathcal{M}_\Delta \models \varphi$ iff $\varphi \in \Delta$.

We apply induction on the size of φ.

Base Case : (1) $\varphi : t_1 = t_2$

(2) $\varphi : P_i^n t_1 t_2 \ldots t_n$

I.H. : Suppose the result holds for all formulas of size $\leq n$.

I.S. : Consider φ s.t. size of φ is $n+1$

(1) $\varphi : \neg\psi$, (2) $\varphi : \psi \vee \chi, \psi \wedge \chi, \psi \to \chi, \psi \leftrightarrow \chi$

(3) $\varphi : \forall x \psi, \exists x \psi$.

Suppose the result holds for the base cases.

Consider Induction Step :

(1) $\varphi : \neg\psi$

$\mathcal{M}_\Delta \models \varphi$ iff $\mathcal{M}_\Delta \models \neg\psi$ iff $\mathcal{M}_\Delta \not\models \psi$ iff $\psi \notin \Delta$ (I.H.) iff $\neg\psi \in \Delta$ iff $\varphi \in \Delta$.

(2) $\varphi : \psi \vee \chi$

$\mathcal{M}_\Delta \models \varphi$ iff $\mathcal{M}_\Delta \models \psi \vee \chi$ iff $\mathcal{M}_\Delta \models \psi$ or $\mathcal{M}_\Delta \models \chi$ iff $\psi \in \Delta$ or $\chi \in \Delta$ (I.H.) iff $\psi \vee \chi \in \Delta$ iff $\varphi \in \Delta$.

H.W. Prove the cases of $\psi \wedge \chi, \psi \to \chi, \psi \leftrightarrow \chi$.

We now need to show the base cases and I.S. (Case 3).

Let us prove the base cases.

We first try to define $\mathcal{M}_\Delta$.

Define $D_\Delta = \{t : t \text{ is a term}\}$

Define I_Δ :

- $I_\Delta(c) = c$ for all $c \in \mathcal{C}$.
- $I_\Delta(f_i^n) : D_\Delta^n \to D_\Delta$ given as follows:

  $I_\Delta(f_i^n)(t_1, t_2, \dots, t_n) = f_i^n t_1 t_2 \cdots t_n$
- $I_\Delta(P_i^n) \subseteq D_\Delta^n$ given as follows:

  $(t_1, t_2, \dots, t_n) \in I(P_i^n)$ iff

  $P_i^n t_1 t_2 \dots t_n \in \Delta$

Define $\mathcal{G}_\Delta$:

$\mathcal{G}_\Delta(x) = x$ for all $x \in V$.

So, $\mathcal{M}_\Delta = (D_\Delta, I_\Delta, \mathcal{G}_\Delta)$.

Now, $\mathcal{G}_\Delta$ can be extended uniquely to $\mathcal{G}_\Delta' : \mathcal{T} \to D_\Delta$, and hence we also mention $\mathcal{G}_\Delta'$ as $\mathcal{G}_\Delta$

H.W. $g_\Delta(t) = t$ for all terms $t \in T$.

Now, we show that $M_\Delta \models \varphi$ iff $\varphi \in \Delta$.

Base cases: (1) $\varphi : t_1 = t_2$

(2) $\varphi : P_i^n t_1 t_2 \cdots t_n$

Let us first prove (2)

$$M_\Delta \models \varphi$$

iff $M_\Delta \models P_i^n t_1 t_2 \cdots t_n$

iff $(g_\Delta(t_1), g_\Delta(t_2), \cdots, g_\Delta(t_n)) \in I_\Delta(P_i^n)$

iff $(t_1, t_2, \cdots, t_n) \in I_\Delta(P_i^n)$

iff $P_i^n t_1 t_2 \cdots t_n \in \Delta$

iff $\varphi \in \Delta$

This completes the proof.

We now move on to Base Case (1).

$$\boxed{\text{To prove : } M_\Delta \models t_1 \equiv t_2 \text{ iff } t_1 \equiv t_2 \in \Delta}$$

Now, $M_\Delta \models t_1 \equiv t_2$ iff $g_\Delta(t_1) = g_\Delta(t_2)$

iff $\underline{t_1 = t_2}$ $\boxed{\text{iff}}$ $t_1 \equiv t_2 \in \Delta$.

$\downarrow$

$t_1 = t_2 = t$??

We are using $\equiv$ for syntactic equality and $=$ for semantic equality.

(in the language of arithmetic :

$1 = 1$, $2 = 2$, - - - - -

but, we would also like to equate

$2+1$ and 3)

Let us go back to our domain definition.

We had $D_\Delta = \{t : t \text{ is a term}\}$

Let us define a relation $\approx$ on $\mathcal{T}$

as follows : $t_1 \approx t_2$ iff $t_1 \equiv t_2 \in \Delta$.

- $\approx$ is an equivalence relation.

Proof: (1) $\approx$ is reflexive.

We have to show $t \equiv t \in \Delta$ for all terms t. Suppose not. There is a term t s.t. $t \equiv t \notin \Delta$. Since Δ is complete, being a model set, $\neg(t \equiv t) \in \Delta$. So $\{\neg(t \equiv t)\}$ is satisfiable, a contradiction.

So, $t \equiv t \in \Delta$ for all terms t.

So, $\approx$ is reflexive.

H.W. (2) $\approx$ is symmetric

(3) $\approx$ is transitive

This completes the proof that $\approx$ is an equivalence relation.

This equivalence relation gives rise to a collection of equivalence classes over D_Δ.

Define $\boxed{D'_\Delta = \{[t] : t \text{ is a term}\}}$, where $[t]$ denotes the equivalence class of t under the equivalence relation defined above.

Let us now define I'_Δ and $\mathcal{G}'_\Delta$.

- $I'_\Delta(c) = [c]$
- $I'_\Delta(f_i^n) : (D'_\Delta)^n \to D'_\Delta$ is defined as follows:

$$I'_\Delta(f_i^n)([t_1],[t_2], \ldots, [t_n]) = [f_i^n t_1 t_2 \cdots t_n]$$

- $I'_\Delta(P_i^n) \subseteq (D'_\Delta)^n$ is defined as follows:

$$([t_1], \ldots [t_n]) \in I'_\Delta(P_i^n) \text{ iff } P_i^n t_1 t_2 \cdots t_n \in \Delta$$

Here, we need to check that $I'_\Delta(f_i^n)$ and $I'_\Delta(P_i^n)$ are well-defined.

- $I'_\Delta(f_i^n)$ is well-defined.

Let $t_1, t_2, \ldots, t_n, t'_1, t'_2, \ldots, t'_n$ be terms s.t. $t_1 \approx t'_1, t_2 \approx t'_2, \ldots, t_n \approx t'_n$. To show, $f_i^n t_1 t_2 \cdots t_n \approx f_i^n t'_1 t'_2 \cdots t'_n$.

That is, we have to show:

If $t_1 \equiv t'_1, t_2 \equiv t'_2, \ldots, t_n \equiv t'_n \in \Delta$, then

$f_i^n t_1 t_2 \cdots t_n \equiv f_i^n t'_1 t'_2 \cdots t'_n \in \Delta$.

Suppose $t_1 \equiv t'_1, t_2 \equiv t'_2, \ldots, t_n \equiv t'_n \in \Delta$

[Lemma: Let φ be a formula. Then, $\models \varphi$ implies $\varphi \in \Delta$ (fin-sat and complete).

Proof: Suppose $\models \varphi$ and $\varphi \notin \Delta$. Then $\neg\varphi \in \Delta$. So, $\{\neg\varphi\}$ is satisfiable, a contradiction. Hence, $\varphi \in \Delta$.]

We have:

$$\models (t_1 \equiv t_1' \wedge t_2 \equiv t_2' \wedge \cdots \wedge t_n \equiv t_n') \rightarrow (f_i^n t_1 t_2 \cdots t_n \equiv f_i^n t_1' t_2' \cdots t_n')$$

H.W. Prove the above validity.

So, by the lemma we just proved,

$(t_1 \equiv t_1' \wedge t_2 \equiv t_2' \wedge \cdots \wedge t_n \equiv t_n') \rightarrow (f_i^n t_1 t_2 \cdots t_n \equiv f_i^n t_1' t_2' \cdots t_n') \in \Delta$.

Now, as $t_1 \equiv t_1'$, $t_2 \equiv t_2'$, ---, $t_n \equiv t_n' \in \Delta$, $t_1 \equiv t_1' \wedge t_2 \equiv t_2' \wedge \cdots \wedge t_n \equiv t_n' \in \Delta$, Δ being a model set. So, we have

$f_i^n t_1 t_2 \cdots t_n \equiv f_i^n t_1' t_2' \cdots t_n' \in \Delta$.

This completes the proof of well-definedness of $I_\Delta'(f_i^n)$.

- $I'_\Delta(P^n_i)$ is well-defined : H.W.

Now, let us define g'_Δ as follows:

$g'_\Delta(x) = [x]$, for all variables $x \in V$.

So, we have our model :

$$\boxed{M'_\Delta = (D'_\Delta, I'_\Delta, g'_\Delta)}$$

As earlier, we can show that,

$g'_\Delta(t) = [t]$ for all terms $t \in T$: H.W.

Let us go back to the result we have to prove:

For all formulas φ, $M'_\Delta \models \varphi$ iff $\varphi \in \Delta$.

Base Case : (1) $t_1 \equiv t_2$: $M'_\Delta \models t_1 \equiv t_2$

iff $g'_\Delta(t_1) = g'_\Delta(t_2)$ iff $[t_1] = [t_2]$ iff

$t_1 \approx t_2$ iff $t_1 \equiv t_2 \in \Delta$.

② $P_i^n t_1 t_2 \cdots t_n$: $\mathcal{M}_\Delta \models P_i^n t_1 t_2 \cdots t_n$ iff $(g'_\Delta(t_1), g'_\Delta(t_2), \cdots, g'_\Delta(t_n)) \in I(P_i^n)$

iff $([t_1], [t_2], \cdots, [t_n]) \in I(P_i^n)$ iff

$P_i^n t_1 t_2 \cdots t_n \in \Delta$.

So, we are done for the base cases.
Now, by our previous argument we are also done for the formulas $\neg\psi$, $\psi \vee \chi$, $\psi \wedge \chi$, $\psi \to \chi$, $\psi \leftrightarrow \chi$. Thus if we consider the atomic formulas (base cases) and their boolean combinations (done earlier), then we have the compactness theorem for the logical language:

$$\varphi, \psi : t_1 = t_2 \mid P_i^n t_1 t_2 \cdots t_n \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \to \psi \mid \varphi \leftrightarrow \psi$$

We call this zeroth order logic