The quantifier cases

Let us now consider the quantified formulas, $\forall x\, \psi$ and $\exists x\, \psi$. We will be done if we can show:

$$\boxed{\begin{array}{l} \mathcal{M}'_\Delta \models \forall x\, \psi \text{ iff } \forall x\, \psi \in \Delta \\ \mathcal{M}'_\Delta \models \exists x\, \psi \text{ iff } \exists x\, \psi \in \Delta \end{array}}$$

How do we proceed?

Now, $\mathcal{M}'_\Delta \models \forall x\, \psi$ iff for all $d \in D'_\Delta$,

$$\mathcal{M}'_{\Delta\,[x \mapsto d]} \models \psi.$$

Here, $D'_\Delta = \{ [t] \mid t \text{ is a term} \}$

What do we mean by a term replacing a variable in a formula?

To answer this question we introduce the concept of 'substitution' as follows:

Substitutions.

(in : Terms)

Let t and t' be two terms and x be a variable. We write $t[t'/x]$ for t' replacing x in the term t.

- $x[t'/x] = t'$
- $y[t'/x] = y, \quad y \neq x$
- $c[t'/x] = c$
- $f^n_i t_1 \cdots t_n [t'/x] = f^n_i t_1[t'/x] \cdots t_n[t'/x]$

(in : Formulas)

Let φ be a formula, t be a term and x be a variable free in φ. Then, we write $\varphi[t/x]$ for x replaced by t in φ.

- $(t_1 \equiv t_2)[t/x] = (t_1[t/x] \equiv t_2[t/x])$
- $P^n_i t_1 \cdots t_n [t/x] = P^n_i t_1[t/x] \cdots t_n[t/x]$

- $\neg\psi[t/x] = \neg(\psi[t/x])$
- $(\psi \vee \chi)[t/x] = \psi[t/x] \vee \chi[t/x]$
- $(\psi \wedge \chi)[t/x] = \psi[t/x] \wedge \chi[t/x]$
- $(\psi \to \chi)[t/x] = \psi[t/x] \to \chi[t/x]$
- $(\psi \leftrightarrow \chi)[t/x] = \psi[t/x] \leftrightarrow \chi[t/x]$
- $$\begin{cases} (\forall x\, \psi)[t/x] = \forall x\, \psi \\ (\forall y\, \psi)[t/x] = \forall y\, (\psi[t/x]), \quad y \neq x \end{cases}$$
- $$\begin{cases} (\exists x\, \psi)[t/x] = \exists x\, \psi \\ (\exists y\, \psi)[t/x] = \exists y\, (\psi[t/x]), \quad y \neq x \end{cases}$$

Examples.

1. $(x \equiv y)[y/x] = y \equiv y$
2. $(x \equiv y)[x/y] = x \equiv x$
3. $\forall x (x \equiv y)[y/x] = \forall x (x \equiv y)$
4. $\forall x (x \equiv y)[x/y] = \forall x (x \equiv x)$

Now, the next question to ask is regarding satisfiability:

When such substituted formulas are satisfied by a model?

Let M be a model and consider a substituted formula $\varphi[t/x]$. What would be a natural way to think about $M \models \varphi[t/x]$?

$$\boxed{M \models \varphi[t/x] \text{ iff } M_{[x \mapsto \ell(t)]} \models \varphi, \quad M = (D, I, \ell)}$$

Will this always hold? NO

$\varphi \quad : \exists y (\neg (y \equiv x))$: satisfiable

$\varphi[y/x] : \exists y (\neg (y \equiv y))$: unsatisfiable

Substitutability

A term t is said to be <u>substitutable</u> for a variable x in a formula φ when the following hold:

- φ is atomic: t is always substitutable for x in φ
- φ is $\neg\psi$: t is substitutable for x in ψ
- φ is $\psi \vee \chi$: t is substitutable for x in ψ and t is substitutable for x in χ.
- φ is $\psi \wedge \chi$: "
- φ is $\psi \to \chi$: "
- φ is $\psi \leftrightarrow \chi$: "

- ϕ is $\forall y \psi$: x does not occur free in ψ

OR

y should not appear in the term t and t is substitutable for x in ψ.

All the discussion started from the attempt to show:

$$M'_\Delta \models \forall x \psi \text{ iff } \forall x \psi \in \Delta$$

We have that for a model M,

$M \models \forall x \psi$ iff for all $d \in D_M$, $M_{[x \to d]} \models \psi$

In our case, d is of the form $[t]$, where t is a term. And, we have seen that $\psi[t/x]$ gives us a formula, where ψ is a formula. Also, we have something like $\mathcal{M} \models \psi[t/x]$ iff $\mathcal{M}_{[x \to g(t)]} \models \psi$, relating satisfaction of such substituted formulas with that of the original formulas. Let us now formalise this idea.

Proposition. Let φ be a formula, x be a variable, t be a term, $\mathcal{M}$ be a model. Then, $\mathcal{M} \models \varphi[t/x]$ iff $\mathcal{M}_{[x \to g(t)]} \models \varphi$, provided t is substitutable for x in φ.

H.W. Prove this proposition.

Let us now give the proof idea of:

$$\boxed{\mathcal{M}_\Delta \models \forall x \psi \quad \text{iff} \quad \forall x \psi \in \Delta .}$$

Proposition: $\models \forall x \varphi \to \varphi[t/x]$, where t is substitutable for x in φ.

The above proposition will ensure the following:

- if $\forall x \varphi \in \Delta$, then $\varphi[t/x] \in \Delta$, for all t substitutable for x in φ.

Also we have, $\models \forall x \varphi \leftrightarrow \neg \exists x \neg \varphi$.

And, similar to above,

- if $\exists x \varphi \in \Delta$, then $\varphi[t/x] \in \Delta$, for some t substitutable for x in φ.

How do these help?

We have to show,

$$\mathcal{M}_\Delta \models \forall x \psi \text{ iff } \forall x \psi \in \Delta.$$

To prove one side of this 'iff' result we use the properties of the formula $\forall x \psi$, and to prove the other side we use the properties of $\exists x \psi$.

An example :

Language of Arithmetic : Let P be a unary predicate symbol. Consider the set $\Gamma = \{ \exists x P(x), \neg P(0) \} \cup$

$$\{ \neg P(S^k(0)) : k \geq 1 \}$$

This is an infinite set of formulas which is fin-sat. This should be, satisfiable by 'compactness theorem'. And, it is. To ensure such a property, we introduce the concept of witness-fulfilled set of formulas.

Witness-fulfilled :

A set of formulas Δ is said to be witness-fulfilled if for every formula of the form $\exists x \varphi$, if $\exists x \varphi \in \Delta$, then $\varphi[t/x] \in \Delta$ for some term t.

Proposition: Every fin-sat, complete set of formulas can be extended to a fin-sat, complete and witness-fulfilled set of formulas.

Then, we will have the final steps of the result: $\mathcal{M}'_{\Delta} \models \forall x \psi$ iff $\forall x \psi \in \Delta$

Now, we go into the proof details.

- Proposition: $\models \forall x \varphi \to \varphi[t/x]$, where, t is substitutable for x in φ.

Proof: Let $\mathcal{M}$ be a model s.t. $\mathcal{M} \models \forall x \varphi$. Then, for any d in $D_{\mathcal{M}}$, $\mathcal{M}_{[x \to d]} \models \varphi$. So, in particular, we have that, $\mathcal{M}_{[x \to I_{\mathcal{M}}(t)]} \models \varphi$. We have t is substitutable for x in φ.

Then, $\mathcal{M} \models \varphi[t/x]$. Hence, $\mathcal{M} \models \forall x \varphi$ implies $\mathcal{M} \models \varphi[t/x]$ for any model $\mathcal{M}$. So, $\models \forall x \varphi \to \varphi[t/x]$, t is substitutable for x in φ. This completes the proof.

The Gödel - Henkin construction

The construction of the term model was introduced by Henkin. The additional technique of getting these witnesses comes from Gödel. So, what we are doing is generally termed as the Gödel - Henkin construction.

Proposition: Any fin-sat, complete set of formulas Γ can be extended to a fin-sat, complete and witness-fulfilled set of formulas

Proof: We note that a constant symbol c not occuring in a formula ϕ can always be substitutable for x in ϕ. We use this idea below.

We expand our language $\mathcal{L}$ with a countable set of new constant symbols; $\mathcal{D}$, say, where $\mathcal{D} = \{d_1, d_2, \ldots\}$

So, the parameters of the new language $\mathcal{L}'$ is given by the parameters of $\mathcal{L}$, the language we started off together with the set $\{d_1, d_2, \dots\}$. In other words, $\mathcal{L} : (\mathcal{C}, \mathcal{F}, \mathcal{P})$ and

$\mathcal{L}' : (\mathcal{C} \cup \mathcal{D}, \mathcal{F}, \mathcal{P})$.

As $\mathcal{L}$ is countable, $\mathcal{L}'$ is also countable. Let us enumerate the formulas in $\mathcal{L}'$ as follows:

$\beta_0, \beta_1, \beta_2, \dots$

We now construct sets of formulas $\Delta_0, \Delta_1, \Delta_2, \dots$, as follows:

- $\Delta_0 = \Gamma$ (fin-sat and complete)
- If $\beta_k = \exists x \varphi$, then,

$$\Delta_{k+1} = \begin{cases} \Delta_k \cup \{\exists x\varphi, \varphi[d/x]\}, & \text{where } d \text{ is the least-indexed constant symbol not occuring in } \Delta_k \cap \beta_k \text{ and the whole set is fin-sat.} \\ \Delta_k, & \text{otherwise} \end{cases}$$

else,

$$\Delta_{k+1} = \begin{cases} \Delta_k \cup \{\beta_k\}, & \text{if it is fin-sat} \\ \Delta_k, & \text{otherwise} \end{cases}$$

We have,

$\Gamma = \Delta_0 \subseteq \Delta_1 \subseteq \Delta_2 \subseteq \text{-----}$

Let $\Delta = \bigcup\limits_{n \geq 0} \Delta_n$

Claim : Δ is fin-sat, complete and witness-fulfilled.

To prove this claim, it is enough to show the following.

Proposition : Let Γ be a fin-sat and complete set of formulas. Let $\exists x \varphi \in \Gamma$. Let c be a constant symbol not occuring in Γ. Then, $\Gamma \cup \{\varphi[c/x]\}$ is fin-sat.

Proof : Suppose not, that is, $\Gamma \cup \{\varphi[c/x]\}$ is not fin-sat. Then, there is $\Gamma' \subseteq_{fin} \Gamma$

s.t. $\Gamma' \cup \{\varphi[c/x]\}$ is unsatisfiable.

So, $\Gamma' \models \neg\varphi[c/x]$. Let $\Gamma' = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$.

Then, $\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \models \neg\varphi[c/x]$. Then,

$\models (\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n) \rightarrow \neg\varphi[c/x]$ H.W.

Now, c does not occur in $\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n$.

Then, $\models (\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n) \rightarrow \forall x \neg\varphi$ (Assume this for now)

Then, $\forall x \neg\varphi \in \Gamma$, a contradiction to the fact that $\exists x \varphi \in \Gamma$. Hence we have our result that $\Gamma \cup \{\varphi[c/x]\}$ is fin. sat.

We now prove the assumption we made above.

Lemma (Generalization of constants):

If $\models \gamma \rightarrow \varphi[c/x]$, where c does not occur in γ and φ, then $\models \gamma \rightarrow \forall x \varphi$.

Proof: Suppose not. Then, there is a model $\mathcal{M}$ s.t. $\mathcal{M} \not\models \gamma \to \forall x \varphi$. Then, $\mathcal{M} \models \gamma$ and $\mathcal{M} \not\models \forall x \varphi$. Now, $\mathcal{M} \not\models \forall x \varphi$ implies $\mathcal{M}_{[x \to d]} \not\models \varphi$ for some d in $D_{\mathcal{M}}$. So, $\mathcal{M}_{[x \to d]} \models \neg \varphi$. Now, let us consider a model $\mathcal{M}'$, same as $\mathcal{M}$, but where $I(c) = d$. Then, $\mathcal{M}'_{[x \to d]} \models \neg \varphi$ (as c does not occur in φ). Also, we have: $\mathcal{M}' \models \gamma$ (because, $\mathcal{M} \models \gamma$ and c does not occur in γ). Then, $\mathcal{M}' \models \varphi[c/x]$. So, $\mathcal{M}'_{[x \to g'(c)]} \models \varphi$, that is, $\mathcal{M}'_{[x \to d]} \models \varphi$. So, we have a contradiction. This completes the proof.

Now, we are almost ready to prove our final step: $\mathcal{M}'_{\Delta} \models \forall x \psi$ iff $\forall x \psi \in \Delta$. We assume Δ to be fin-sat, complete and witness-fulfilled.

Proof of $\mathcal{M}'_\Delta \models \forall x \psi$ iff $\forall x \psi \in \Delta$ [$\mathcal{M}'_\Delta$ is a model in language $\mathcal{L}'$ now]

- Suppose $\mathcal{M}'_\Delta \models \forall x \psi$. To show $\forall x \psi \in \Delta$. Suppose not. That is, $\forall x \psi \notin \Delta$. Then, $\neg \forall x \psi \in \Delta$. Then, $\exists x \neg \psi \in \Delta$. So, $\neg \psi[t/x] \in \Delta$ for some term t. So, $\psi[t/x] \notin \Delta$. So, $\mathcal{M}'_\Delta \not\models \psi[t/x]$ (by I.H.) Then, $\mathcal{M}'_{\Delta\,[x \to \mathcal{I}'_\Delta(t)]} \not\models \psi$. So, $\mathcal{M}'_{\Delta\,[x \to [t]]} \not\models \psi$. So, $\mathcal{M}'_\Delta \not\models \forall x \psi$, a contradiction. So, we have: $\forall x \psi \in \Delta$. This completes the proof.

- Conversely, suppose that $\forall x \psi \in \Delta$. To prove, $\mathcal{M}'_\Delta \models \forall x \psi$. So, to prove, $\mathcal{M}'_{\Delta\,[x \to [t]]} \models \psi$ for all terms t. Suppose not. Then, $\mathcal{M}'_{\Delta\,[x \to [t]]} \not\models \psi$ for some term t. If t is substitutable for x in ψ, we have, $\mathcal{M}'_\Delta \not\models \psi[t/x]$. So, $\mathcal{M}'_\Delta \models \neg \psi[t/x]$

Then, $\neg\psi[t/x] \in \Delta$ (I.H.). But as, $\forall x\, \psi \in \Delta$, $\psi[t/x] \in \Delta$ (why?). We have a contradiction, hence the result.

But, we need the result for all terms t. What do we do? We prove the following lemma.

Lemma: Given any formula φ, variable x and term t, there exists a formula φ' (alphabetic variant of φ), obtained by renaming of bound variables in φ, s.t. t is substitutable for x in φ' and $\models \varphi \leftrightarrow \varphi'$.

H.W. Prove this lemma.

Let us now go back to the converse proof.

Suppose $\forall x\, \psi \in \Delta$. To prove $\mathcal{M}'_\Delta \models \forall x\, \psi$

Suppose not. Then, $\mathcal{M}'_\Delta \not\models \forall x\, \psi$. Then,

$\mathcal{M}'_{\Delta\,[x \mapsto [t]]} \not\models \psi$ for some term t.

So, $\mathcal{M}'_{\Delta\,[x \mapsto [t]]} \not\models \psi'$ (an alphabetic variant of ψ)

So, $\mathcal{M}'_\Delta \not\models \psi'[t/x]$ (since t is substitutable for x in ψ')

So, $\psi'[t/x] \notin \Delta$ (I. H.)

So, $\forall x\, \psi' \notin \Delta$ (since $\models \forall x\, \varphi \to \varphi[t/x]$, where t is substitutable for x in φ)

$\forall x\, \psi \notin \Delta$, a contradiction.

$\models \varphi \leftrightarrow \varphi'$ implies $\models \forall x\, \varphi \leftrightarrow \forall x\, \varphi'$

H.W. Prove the above statement.

This completes the proof.

Thus we have finished the proof of the statement: for all formulas φ, $\mathcal{M}'_\Delta \models \varphi$ iff $\varphi \in \Delta$.

This statement is generally termed as the 'truth lemma'.

Let us recapitulate the proof ideas that we have considered:

1. Start with a fin-sat set Γ.

2. Extend the set to form a fin-sat and complete set.

3. Show that the extended set becomes a model set.

4. Then show that this set has a model. This gave the proof of the truth lemma for quantifier-free formulas (zero-th order logic).

5. Extend the fin-sat and complete set to a fin-sat, complete and witness-fulfilled set to get the proof of the truth lemma for the quantified formulas.

While doing all these we extended the the language from L to L'. Thus the model we constructed is a model for Γ in the language $\mathcal{L}'$. But, we need to have a model for Γ in the language $\mathcal{L}$, where $\mathcal{L} : (\mathcal{C}, \mathcal{F}, \mathcal{P})$, $\mathcal{L}' : (\mathcal{C} \cup \mathcal{D}, \mathcal{F}, \mathcal{P})$.

<u>How to get that required model?</u>

Let $\mathcal{M}'$ be a model for the language $\mathcal{L}'$, that is, $\mathcal{M}' = (D', I', g')$. Here,

$$I' \begin{cases} \mathcal{C} \cup \mathcal{D} \\ \mathcal{F} \\ \mathcal{P}. \end{cases}$$

Now, we can restrict $\mathcal{M}'|_{\mathcal{L}}$ given by,

$\mathcal{M} = (D, I, g)$, where,

$$D = D'$$
$$g = g'$$
$$I \begin{cases} \mathcal{C} \\ \mathcal{F}. \\ \mathcal{P} \end{cases}$$

Now, Γ is a set of formulas in $\mathcal{L}$, so, it does not contain any constant symbol from D. In the proof earlier we constructed a model M'_Δ for Γ in the language $\mathcal{L}'$.

However, as no symbol from D occur in Γ, $M'_\Delta\big|_{\mathcal{L}}$ forms a model of Γ as well. And, $M'_\Delta\big|_{\mathcal{L}}$ is a model corresponding to the language $\mathcal{L}$. Thus, we get a model of Γ in the language $\mathcal{L}$. Hence, we have that Γ is satisfiable.

Thus, every fin-sat set of formulas is sat. This marks the end of the proof of compactness theorem.