Compactness Theorem of F.O.L

Let Γ be an infinite set of formulas. Then, Γ is sat iff every finite subset of Γ is sat (Γ is fin-sat).

Non-trivial part: If Γ is fin-sat, then Γ is sat.

How does this result connect with the consequence relation?

① If Γ is fin-sat then Γ is sat.

② If $\Gamma \models \varphi$, then there is a finite subset Γ_{fin} of Γ s.t. $\Gamma_{fin} \models \varphi$.

Proposition: ① iff ②

Proof: ② ⇒ ①: Let Γ be fin-sat. To show that Γ is sat. Suppose not. Then, $\Gamma \models \varphi$ for all formulas φ. Then there is a formula ψ, say, s.t. $\Gamma \models \psi$ and $\Gamma \models \neg\psi$. Then, there are $\Gamma_1, \Gamma_2 \subseteq_{fin} \Gamma$ s.t. $\Gamma_1 \models \psi$ and $\Gamma_2 \models \neg\psi$. (by ②). So,

$\Gamma_1 \cup \Gamma_2 \models \psi \wedge \neg\psi$ and $\Gamma_1 \cup \Gamma_2 \subseteq_{fin} \Gamma$. Thus, $\Gamma_1 \cup \Gamma_2$ is unsatisfiable. This contradicts the assumption that Γ is fin-sat. This completes the proof.

(1) $\Rightarrow$ (2): Let $\Gamma \models \varphi$. To show that there is $\Gamma_f \subseteq_{fin} \Gamma$ s.t. $\Gamma_f \models \varphi$. Suppose not. So, for all $\Gamma' \subseteq_{fin} \Gamma$, $\Gamma' \not\models \varphi$. So, for all $\Gamma' \subseteq_{fin} \Gamma$, $\Gamma' \cup \{\neg\varphi\}$ is sat. So, by (i) $\Gamma \cup \{\neg\varphi\}$ is sat. Then, $\Gamma \not\models \varphi$, a contradiction. This completes the proof

More applications of Compactness Theorem.

1. Let Σ be a set of sentences having arbitrarily large finite models. Then, Σ has an infinite model.

Proof: Let $\mathcal{D} = \{d_1, d_2, \ldots\}$ be a countable collection of new constant symbols not occurring in Σ. Consider $\Delta = \Sigma \cup \{\neg(d_i \equiv d_j) : i, j \in \mathbb{N}, i \neq j\}$. Now, Σ is sat. So, Σ is fin-sat. Take any finite subset of $\{\neg(d_i \equiv d_j) : i, j \in \mathbb{N}, i \neq j\}$. Such a finite subset will be satisfiable in some model of Σ. So, we have that Δ is fin-sat. So, by compactness theorem Δ is sat. But, a model of Δ will contain infinitely many elements. Now, $\Sigma \subseteq \Delta$. So any model of Δ is a model of Σ. Thus, Σ has an infinite model. This completes the proof.

A weaker notion of definability

A class K of structures is said to be definable in FOL in the weaker sense if there is a set of sentences, Σ, say, s.t. $K = \{ S : S \models \Sigma \}$, or, in other words, $K = Mod(\Sigma)$

2. Let FIN denote the class of all finite structures. FIN is not fo-definable, not even in the weaker sense.

H.W. Prove this statement.

3. Downward Lowenheim-Skolem Theorem: Let Γ be a satisfiable set of formulas. Then Γ has a countable model.

Proof idea: Since Γ is sat, Γ is fin-sat. Using this fin-sat assumption, we can come up with a countable model of Γ through a proof of compactness theorem.

Important fact: First-order language is countable, in other words, the set of formulas is countable.

Consider the two questions posed earlier:

1. Is the class of infinite sets definable in FOL?
2. How to show that elementary equivalence does not imply isomorphism?

Answer to 1.

The answer is NO. Had the class of infinite models, $\mathcal{I}$, say, be definable, $\mathcal{I} = Mod(\tau)$, where τ is a first-order sentence. Then, FIN, the class of all finite structures could be written as $FIN = Mod(\neg\tau)$. This is not possible by Application (2) above. Hence the claim.

Answer to 2

We need to consider two structures and show that they are elementarily equivalent but not isomorphic. Consider an fo-language $\mathcal{L}_<$, say, with a single parameter $<$, a binary relation symbol. Consider two structures $(\mathbb{R}, <)$ and $(\mathbb{Q}, <)$. Of course, they are not isomorphic. We will now show that $(\mathbb{R}, <)$ and $(\mathbb{Q}, <)$ are elementarily equivalent. To prove this, we will use the following statement:

Any two countable dense linear orders without end-points are isomorphic. (Cantor)

(A good exercise to try out!)

Theories

A first-order theory is a set of sentences T s.t for all sentences σ, if $T \models \sigma$ then $\sigma \in T$. In other words, $T = \mathrm{Con}(T)$.

Complete theories

A theory T is said to be complete if for any sentence σ, either $\sigma \in T$ or, $\neg\sigma \in T$.

Proposition: A theory T is complete iff any two models of T are elementarily equivalent.

H.W.: Prove this proposition.

$\aleph_0$-categorical theory

A theory T is said to be $\aleph_0$-categorical if all its models of cardinality $\aleph_0$ are isomorphic.

Łos-Vaught test: Let T be theory. If T is $\aleph_0$-categorical, then T is complete.

Proof: Let T be an $\aleph_0$-categorical theory. To show that T is complete. It is enough to show that any two models of T, $\mathcal{A}$ and $\mathcal{B}$, say, are elementarily equivalent (denote by $\mathcal{A} \equiv \mathcal{B}$).

For the model $\mathcal{A}$, consider $\Sigma_{\mathcal{A}} = \{\sigma : \mathcal{A} \models \sigma\}$.

Then, $\Sigma_{\mathcal{A}}$ is a satisfiable set of sentences

Then, by Downward Lowenheim Skolem Theorem, $\Sigma_{\mathcal{A}}$ has a countable model, $\mathcal{A}'$, say. Then, $\mathcal{A} \equiv \mathcal{A}'$. Additionally, $\mathcal{A}'$ is a model of T, as $T \subseteq \Sigma_{\mathcal{A}}$. Similarly, we get a countable model $\mathcal{B}'$ of T s.t. $\mathcal{B} \equiv \mathcal{B}'$. So, $\mathcal{A}'$ and $\mathcal{B}'$ are isomorphic. So, $\mathcal{A}' \cong \mathcal{B}'$. Then: $\mathcal{A} \equiv \mathcal{A}' \equiv \mathcal{B}' \equiv \mathcal{B}$. Since $\mathcal{A}$ and $\mathcal{B}$ are two arbitrary models of T, we have that T is complete. This completes the proof.

$$(\mathbb{R}, <_{\mathbb{R}}) \equiv (\mathbb{Q}, <_{\mathbb{Q}})$$

Proof: Let $\mathcal{L}_<$ be the language of order: $<$. Consider the following set of sentences.

1. (a) $\forall x \forall y (x < y \vee x = y \vee y < x)$

   (b) $\forall x \forall y (x < y \to \neg (y < x))$

   (c) $\forall x \forall y \forall z (x < y \to (y < z \to x < z))$

2. $\forall x \forall y (x < y \to \exists z (x < z \wedge z < y))$

3. $\forall x \exists y \exists z (y < x \wedge x < z)$

S

Let $T = \mathrm{Con}(S)$ is a first order theory whose models are $(\mathbb{R}, <_{\mathbb{R}})$ and $(\mathbb{Q}, <_{\mathbb{Q}})$. Since T is the theory of dense linear orders without end-points, from Cantor's theorem, T is $\aleph_0$-categorical. So, by Łoś-Vaught Test, T is complete. So, all models of T are elementarily-equivalent. Hence, $(\mathbb{R}, <_{\mathbb{R}}) \equiv (\mathbb{Q}, <_{\mathbb{Q}})$. This completes the proof.

We now move on to the analysis of the mathematical reasoning that we do while trying to prove various results. To study this formally, we introduce the following notion:

(Deductive) Consequence Relation

Γ: a set of formulas; φ: a formula.

$\Gamma \vdash \varphi$: φ is a deductive consequence of Γ.

Deductive consequence relations can be defined in various ways: e.g., Hilbert-style axiom systems, Gentzen's Sequent Calculus, Gentzen's Natural Deduction. For this course, we will focus on the first one.

Definition of $\Gamma \vdash \varphi$ with Hilbert-style axiomatization

Let Γ be a set of formulas and φ be a formula. φ is said to be a deductive consequence of Γ ($\Gamma \vdash \varphi$) if there is a sequence of formulas, $\varphi_1, \varphi_2, \ldots, \varphi_n$, such that:

- φ_n is φ
- each φ_i is either an axiom or a member of Γ or obtained by the application of some rule of inference.

What are these axioms and what are the rules?

- Axioms are formulas in the language, $\mathcal{L}$, say.
- A rule of inference is a subset of $\wp(\mathcal{L}) \times \mathcal{L}$.

which we write as follows: $\dfrac{\psi_1, \psi_2, \ldots, \psi_n}{\psi}$,

where $\{\psi_1, \psi_2, \ldots, \psi_n\}$ constitute the premise of the rule and ψ is the consequence of the rule.

How are the two consequence relations related?

$$\Gamma \models \varphi \quad \overset{\text{soundness}}{\underset{\text{completeness}}{\Longleftrightarrow}} \quad \Gamma \vdash \varphi$$

Soundness: If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.

Completeness: If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

Let us focus on soundness. $\Gamma \vdash \varphi \Rightarrow \Gamma \models \varphi$

What kind of properties would we like to prove for these axioms and rules to get our soundness result?

Let us explore. Suppose $\Gamma \vdash \varphi$. Then, we have

$\Gamma \vdash \varphi_1$ ——— To prove: $\Gamma \models \varphi_1$

$\Gamma \vdash \varphi_2$ ——— To prove: $\Gamma \models \varphi_2$

$\vdots$

$\Gamma \vdash \varphi_n \, (= \varphi)$ ——— To prove: $\Gamma \models \varphi_n \, (= \varphi)$.

- The above suggests that an approach to prove soundness could be to use induction on the length n of the derivation sequence $\varphi_1, \cdots, \varphi_n$ of φ from Γ. Such a sequence is also termed as a proof of φ from Γ. In such a derivation sequence or a proof, any formula is either an axiom or a member of Γ or obtained by some rule.

What do we mean by 'obtained by a rule'?

An application of a rule basically means that when $\psi_1, \psi_2, \cdots, \psi_k$ appear in some sequence, $\varphi_1, \varphi_2, \cdots, \varphi_n$, say, ψ can be written as a next step in the sequence. And we have an application of the rule: $\dfrac{\psi_1, \psi_2, \cdots, \psi_k}{\psi}$.