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Soundness theorem: If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$

Proof: Suppose $\Gamma \vdash \varphi$. To show $\Gamma \models \varphi$. We apply induction on the length of proof of φ from Γ.

Base Case: When $n = 1$, $\varphi_n \in \Gamma$ or, φ_n is an axiom.

- if $\varphi_n \in \Gamma$, of course, $\Gamma \models \varphi_n$
- if φ_n is an axiom, to show $\Gamma \models \varphi_n$, we have to show that all models satisfy φ_n, in other words, φ_n is a validity.

* We need to check that all axioms are validities.

Induction hypothesis: Suppose the result holds for all proofs of length $\leq m$.

Induction step: Suppose $n = m+1$, $m \geq 1$. Then, φ_n is (i) a member of Γ, or, (ii) an axiom, or, (iii) obtained by some rule.

Cases (i) and (ii) can be dealt with as earlier.

Case (iii) We have to show that for any model satisfying the premises of a rule, the model should also satisfy the consequence of the rule.

* We need to check that the rules preserve consequence.

Thus to prove soundness, we show: (i) Axioms are validities. (ii) Rules preserve consequence. The proof follows.

Remark: When $\vdash \varphi$, we say: φ is a theorem. Then, the sequence $\varphi_1, \varphi_2, \ldots, \varphi_n$ of formulas would either be axioms or obtained by rules.

Classical Propositional Logic (CPL)

Alphabet : p, $|$, $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$, (,) . Abbreviation : p followed by n vertical lines $|$ is denoted by p_n.

Let P denote a countable set of these propositional variables p_i.

Language ($\mathcal{L}$) :

$$\alpha, \beta := \underset{\downarrow}{p_n} \,|\, (\neg \alpha) \,|\, (\alpha \vee \beta) \,|\, (\alpha \wedge \beta) \,|\, (\alpha \rightarrow \beta) \,|\, (\alpha \leftrightarrow \beta).$$

$$p_n \in P$$

Model :

$V : P \rightarrow \{0,1\}$ (valuation)

Any $V : P \rightarrow \{0,1\}$ can be uniquely extended to $V' : \mathcal{L} \rightarrow \{0,1\}$ with respect to the truth tables for the connectives.

A formula α is said to be valid if $V(\alpha) = 1$ for all V's.

Valid formulas (Tautologies) :

1. $\alpha \rightarrow \alpha$.
2. $(\alpha \wedge \beta) \rightarrow \alpha$
3. $\alpha \rightarrow (\alpha \vee \beta)$

Consequence preserving rules

1. $\dfrac{\alpha, \ \alpha \rightarrow \beta}{\beta}$ 2. $\dfrac{\alpha \vee \beta \qquad \neg\alpha \vee \gamma}{\beta \vee \gamma}$

Task : Prove $\Gamma \vdash \alpha$ iff $\Gamma \models \alpha$ (Soundness & Completeness)

But, what are the axioms and rules of CPL ?

Let us now focus on the completeness proof to answer this question above, that is, we proceed to prove : $\Gamma \models \alpha$ implies $\Gamma \vdash \alpha$
These are two entirely different concepts. We have studied $\models$ quite well. Let us see how we can prove some properties of $\models$ with respect to $\vdash$.

Property 1 : If $\Gamma \models \alpha$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \models \alpha$.
To show : If $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \alpha$. It follows from the definition of $\vdash$

Property 2 : $\{\alpha_1, \cdots, \alpha_n\} \models \beta$ iff $\models (\alpha_1 \wedge \cdots \wedge \alpha_n) \to \beta$.
In general, $\Gamma \cup \{\alpha\} \models \beta$ iff $\Gamma \models \alpha \to \beta$.
To show : $\Gamma \cup \{\alpha\} \vdash \beta$ iff $\Gamma \vdash \alpha \to \beta$. (Deduction Theorem)

Proof : (if - part) Suppose $\Gamma \vdash \alpha \to \beta$. To show : $\Gamma \cup \{\alpha\} \vdash \beta$.
Now, $\Gamma \cup \{\alpha\} \vdash$ 1. α (Premise)
2. $\alpha \to \beta$ (Assumption : $\Gamma \vdash \alpha \to \beta$)

[Consider the rule : $\dfrac{\alpha \quad \alpha \to \beta}{\beta}$ (Modus Ponens (M.P.))]

3. β (M.P. 1, 2)

This completes the 'if'-part.

(only 'if'-part) Suppose $\Gamma \cup \{\alpha\} \vdash \beta$. To show : $\Gamma \vdash \alpha \to \beta$
We prove this by applying induction on the length of a derivation / proof of β from $\Gamma \cup \{\alpha\}$.

Base Case $(n=1)$: β is either a member of $\Gamma \cup \{\alpha\}$ or, β is an axiom. Now we have 3 cases :

Case 1 : $\beta = \alpha$. Then we have to show $\Gamma \vdash \alpha \to \alpha$

[Consider an axiom : $\alpha \to \alpha$]

We have $\Gamma \vdash$ 1. $\alpha \to \alpha$.

Case 2 : $\beta \in \Gamma$. To show $\Gamma \vdash \alpha \to \beta$.

Since $\beta \in \Gamma$, we have : $\Gamma \vdash$ 1. β (Premise)

[Consider an axiom : $\beta \to (\alpha \to \beta)$]

2. $\beta \to (\alpha \to \beta)$ (Axiom)

3. $\alpha \to \beta$ (M.P. 1, 2)

Case 3 : β is an axiom. To show $\Gamma \vdash \alpha \to \beta$.

We have the same derivation as above : $\Gamma \vdash$ 1. β

2. $\beta \to (\alpha \to \beta)$

3. $\alpha \to \beta$.

So, we are done with the base cases.

[Axioms and Rule used : (A1) $\alpha \to \alpha$ (A2) $\alpha \to (\beta \to \alpha)$]

(R1) $\dfrac{\alpha \quad \alpha \to \beta}{\beta}$

Induction Hypothesis : Suppose the result holds for all proofs of length $\leq m$.

$\boxed{\text{If } \Gamma \cup \{\alpha\} \vdash \beta \text{, then } \Gamma \vdash \alpha \to \beta}$

$\Gamma \vdash \alpha \to \delta$

$\Gamma \vdash \vdots \; \left.\begin{matrix} \delta \\ \delta \to \beta \end{matrix}\right\} \leq m$

$\beta \longrightarrow m+1$

Induction Step $(n = m+1)$: By definition of $\vdash$, β is either an axiom or, a member of $\Gamma \cup \{\alpha\}$ or, obtained by some rule. We only deal with the third possibility. Assume that β is obtained by an application of M.P. So, we have : $\Gamma \cup \{\alpha\} \vdash \delta$ and $\Gamma \cup \{\alpha\} \vdash \delta \to \beta$, from which we have, by M.P., $\Gamma \cup \{\alpha\} \vdash \beta$. To show that $\Gamma \vdash \alpha \to \beta$. By I.H., we have:

$\Gamma \vdash$ 1. $\alpha \to \delta$ (I.H.)

2. $\alpha \to (\delta \to \beta)$ (I.H.)

[Consider an axiom: $(\alpha \to (\delta \to \beta)) \to ((\alpha \to \delta) \to (\alpha \to \beta))$]

3. $(\alpha \to (\delta \to \beta)) \to ((\alpha \to \delta) \to (\alpha \to \beta))$ (Axiom)

4. $(\alpha \to \delta) \to (\alpha \to \beta)$ (M.P. 2,3)

5. $\alpha \to \beta$ (M.P. 1,4)

So, we have a derivation of $\alpha \to \beta$ from Γ. This completes the proof of deduction theorem.

Property 3: $\{\alpha, \neg\alpha\} \models \beta$ for any β.

To show: $\{\alpha, \neg\alpha\} \vdash \beta$, for any β.

Now, $\{\alpha, \neg\alpha\} \vdash$ 1. α (Premise)

2. $\neg\alpha$ (Premise)

3. $\alpha \to (\neg\beta \to \alpha)$ (Axiom)

4. $\neg\beta \to \alpha$ (M.P. 1,3)

5. $\neg\alpha \to (\neg\beta \to \neg\alpha)$ (Axiom)

6. $\neg\beta \to \neg\alpha$ (M.P. 2,5)

[Consider an axiom: $(\neg\beta \to \alpha) \to ((\neg\beta \to \neg\alpha) \to \beta)$]

7. $(\neg\beta \to \alpha) \to ((\neg\beta \to \neg\alpha) \to \beta)$ (Axiom)

8. $(\neg\beta \to \neg\alpha) \to \beta$ (M.P. 4, 7)

9. β (M.P. 6, 8)

This completes the proof of $\{\alpha, \neg\alpha\} \vdash \beta$.

Remark! $\alpha \to \alpha$ can be derived from the following axioms:
(A1) $\alpha \to (\beta \to \alpha)$; (A2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$;
and the M.P. rule.

H.W. Find a proof of $\alpha \to \alpha$ using (A1), (A2) and M.P.

Thus we have the following collection of axioms and rules till now.

Axioms : (A1) $\alpha \to (\beta \to \alpha)$

(A2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

(A3) $(\neg\beta \to \alpha) \to ((\neg\beta \to \neg\alpha) \to \beta)$

Rule : $$\frac{\alpha \quad \alpha \to \beta}{\beta} \quad \text{(M.P.)}$$

We will now show in details that these 3 axioms and 1 rule provide a sound and complete axiom system for CPL, that is we need to prove the soundness and completeness theorems, respectively.

To show soundness we just need to prove the the axioms (1), (2) and (3) are valid and the rule M.P. preserves consequence.

H.W. Prove these results

Thus, soundness proof is done once you do your homework.

Completeness Theorem: If $\Gamma \models \alpha$ then $\Gamma \vdash \alpha$.

Proof: We will show that $\Gamma \nvdash \alpha$ implies $\Gamma \not\models \alpha$. Showing $\Gamma \not\models \alpha$ is the same as showing $\Gamma \cup \{\neg\alpha\}$ is satisfiable. So, assuming $\Gamma \nvdash \alpha$, if we can show that $\Gamma \cup \{\neg\alpha\}$ is satisfiable, we are done, that is, completeness would be shown.

How do we show this?

Step 1: We introduce a notion "consistency".

Step 2: We show that $\Gamma \nvdash \alpha$ iff $\Gamma \cup \{\neg\alpha\}$ is consistent.

Step 3: Every consistent set is satisfiable, that is, has a model.

1. Notion of consistency.

- A set of formulas Γ is said to be inconsistent if there is a formula α such that $\Gamma \vdash \alpha$ and $\Gamma \vdash \neg\alpha$.

- A set of formulas Γ is said to be consistent if it is not inconsistent.

2. $\Gamma \nvdash \alpha$ iff $\Gamma \cup \{\neg\alpha\}$ is consistent

if-part: Let $\Gamma \cup \{\neg\alpha\}$ be consistent. To show $\Gamma \nvdash \alpha$. Suppose not. Then $\Gamma \vdash \alpha$. Then, $\Gamma \cup \{\neg\alpha\} \vdash \alpha$. But $\Gamma \cup \{\neg\alpha\} \vdash \neg\alpha$. So, $\Gamma \cup \{\neg\alpha\}$ is inconsistent, a contradiction.

only-if part: Let $\Gamma \nvdash \alpha$. To show: $\Gamma \cup \{\neg\alpha\}$ is consistent. Suppose not. Then, for some formula β, $\Gamma \cup \{\neg\alpha\} \vdash \beta$ and $\Gamma \cup \{\neg\alpha\} \vdash \neg\beta$. So, we have:

$$
\begin{array}{lll}
\Gamma \vdash & 1.\ \neg\alpha \to \beta & (\text{D.T.}) \\
& 2.\ \neg\alpha \to \neg\beta & (\text{D.T.}) \\
& 3.\ (\neg\alpha \to \beta) \to ((\neg\alpha \to \neg\beta) \to \alpha) & (\text{Axiom } 3) \\
& 4.\ (\neg\alpha \to \neg\beta) \to \alpha & (\text{M.P. } 1,3) \\
& 5.\ \alpha & (\text{M.P. } 2,4)
\end{array}
$$

Thus, $\Gamma \vdash \alpha$, a contradiction. Thus, $\Gamma \cup \{\neg\alpha\}$ is consistent. This completes the proof.

3. Every consistent set is satisfiable.

Let Γ be a consistent set of formulas. To show that Γ is satisfiable, that is, to find a valuation $V : P \to \{0,1\}$ such that $V(\gamma) = 1$ for all $\gamma \in \Gamma$. We will show this via the following steps.

(a) Extend the consistent set Γ to a consistent and complete set, Δ, say.

[A set of formulas Δ is said to be complete if for any formula α, either $\alpha \in \Delta$ or, $\neg\alpha \in \Delta$.]

(b) Define a valuation V_Δ, say.

(c) Prove that for all formulas α, $V_\Delta \models \alpha$ ($V_\Delta(\alpha) = 1$) iff $\alpha \in \Delta$. (Truth lemma)

Proof of 3(a): Let Γ be a consistent set of formulas. Let us now enumerate the CPL formulas

$\alpha_0, \alpha_1, \alpha_2, \ldots\ldots$

Define $\Delta_0 = \Gamma$

$$\Delta_{k+1} = \begin{cases} \Delta_k \cup \{\alpha_k\} & \text{if it is consistent} \\ \Delta_k, & \text{otherwise} \end{cases}$$

$$\Delta = \bigcup_{n \in \mathbb{N}} \Delta_n .$$

Claim: Δ is consistent and complete.

It is enough to show the following lemma:

Lemma: If Π is a consistent set of formulas, then either $\Pi \cup \{\alpha\}$ is consistent or, $\Pi \cup \{\neg\alpha\}$ is consistent for any formula α.

Proof: Suppose not. Let β be a formula such that both $\Pi \cup \{\beta\}$ and $\Pi \cup \{\neg\beta\}$ are both inconsistent. Since $\Pi \cup \{\beta\}$ is inconsistent, there is a formula δ such that $\Pi \cup \{\beta\} \vdash \delta$ and $\Pi \cup \{\beta\} \vdash \neg\delta$. So, by D.T., we have: $\Pi \vdash \beta \to \delta$ and $\Pi \vdash \beta \to \neg\delta$. Now, by the axioms and rules that we mentioned earlier, we can show:

$$\vdash (\beta \to \delta) \to ((\beta \to \neg\delta) \to \neg\beta) \quad \text{(Theorem of CPL)}$$

H.W. Prove in CPL: $\vdash (\beta \to \delta) \to ((\beta \to \neg\delta) \to \neg\beta)$

Now we have the following derivation.

$\Pi \vdash$
1. $\beta \to \delta$ (D.T.)
2. $\beta \to \neg\delta$ (D.T.)
3. $(\beta \to \delta) \to ((\beta \to \neg\delta) \to \neg\beta)$ (Theorem)
4. $(\beta \to \neg\delta) \to \neg\beta$ (M.P. 1, 3)
5. $\neg\beta$ (M.P. 2, 4)

So we have: $\Pi \vdash \neg\beta$. Similarly, using the assumption that $\Pi \cup \{\neg\beta\}$ is inconsistent, we can show that $\Pi \vdash \beta$.

H.W. Prove the above statement.

So, we have that $\Pi \vdash \neg\beta$ and $\Pi \vdash \beta$, that is, Π is inconsistent, a contradiction to the fact that Π is consistent. Hence, the result.

- Using this lemma, by the construction of Δ, we can show that Δ is consistent and complete. This completes Step 3.(a).

Definition for 3.(b).

Let Δ be a consistent and complete set of formulas extending the consistent set Γ of formulas we started with. Define:

$$V_\Delta : P \to \{0,1\} \text{ by } V_\Delta(p) = 1 \text{ iff } p \in \Delta.$$