Definition for 3.(b).

Let Δ be a consistent and complete set of formulas extending the consistent set Γ of formulas we started with. Define :

$$V_\Delta : P \to \{0,1\} \text{ by } V_\Delta(p) = 1 \text{ iff } p \in \Delta.$$

Proof of 3(c) [Truth lemma]

To show: for all formulas α, $V_\Delta(\alpha) = 1$ iff $\alpha \in \Delta$.

Before going into the proof let us introduce :

Model Set

A set of formulas Π is said to be a model set if the following holds :

1. (a) For any formula α, $\alpha \in \Pi$ or $\neg\alpha \in \Pi$

(b) Π does not contain both.

2. For any formulas α and β,
(a) $\alpha \vee \beta \in \Pi$ iff ($\alpha \in \Pi$ or, $\beta \in \Pi$)
(b) $\alpha \wedge \beta \in \Pi$ iff ($\alpha \in \Pi$ and, $\beta \in \Pi$)
(c) $\alpha \to \beta \in \Pi$ iff ($\beta \in \Pi$ whenever $\alpha \in \Pi$)
(d) $\alpha \leftrightarrow \beta \in \Pi$ iff ($\alpha \in \Pi$ iff $\beta \in \Pi$).

H.W. Prove that every consistent and complete set of formulas is a model set.

Now, back to proof of 3.(c).

To show: $V_\Delta(\alpha) = 1$ iff $\alpha \in \Delta$. We prove this by applying induction on the size of the formula.

Base case: $\alpha := p$. To show that $V_\Delta(p) = 1$ iff $p \in \Delta$.
This follows from the definition of V_Δ.

I.H.: Suppose the result holds for all formulas of size $\leq m$.
I.S.: Let the size of the formula α be $m+1$.

Case 1: $\alpha := \neg\beta$. $V_\Delta(\alpha) = 1$ iff $V_\Delta(\neg\beta) = 1$ iff $V_\Delta(\beta) = 0$ iff $\beta \notin \Delta$ (by I.H.) iff $\neg\beta \in \Delta$ (Δ is a model set) iff $\alpha \in \Delta$.

Case 2: $\alpha := \beta \to \delta$. $V_\Delta(\alpha) = 1$ iff $V_\Delta(\beta \to \delta) = 1$ iff $V_\Delta(\beta) = 0$ or, $V_\Delta(\delta) = 1$, iff $\beta \notin \Delta$ or, $\delta \in \Delta$ (by I.H.) iff $\delta \in \Delta$ whenever $\beta \in \Delta$ iff $\beta \to \delta \in \Delta$ iff $\alpha \in \Delta$.

This completes the proof of completeness theorem.

Finally, let us list the axioms and rules that we have used to prove completeness. This gives us the axiom-system for CPL.

Axioms

1. $\alpha \to (\beta \to \alpha)$
2. $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
3. $(\neg\alpha \to \beta) \to ((\neg\alpha \to \neg\beta) \to \alpha)$

Rule

$$\frac{\alpha \quad \alpha \to \beta}{\beta}$$

And, we have soundness and completeness for CPL:

$$\Gamma \vdash \alpha \text{ iff } \Gamma \vDash \alpha .$$

This result is termed as the general completeness result. We also have a weak completeness result for CPL: $\vdash \alpha$ iff $\vDash \alpha$

$\Downarrow$ theorem $\qquad$ $\Downarrow$ validity

H.W. Prove the weak completeness result for CPL, independent of the general result.

Theorems in CPL.

1. $\vdash \alpha \to \alpha$.

Proof (a): $\{\alpha\} \vdash \alpha$

$\vdash \alpha \to \alpha$ (D.T.).

Proof (b): 1. $\alpha \to ((\alpha \to \alpha) \to \alpha)$ Axiom 1.

2. $(\alpha \to ((\alpha \to \alpha) \to \alpha)) \to ((\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha))$ Axiom 2

3. $(\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha)$ (M.P. 1, 2)

4. $\alpha \to (\alpha \to \alpha)$ Axiom 1.

5. $\alpha \to \alpha$ (M.P. 3, 4).

2. $\vdash \neg\neg\alpha \to \alpha$

Proof: $\neg\neg\alpha \to (\neg\alpha \to \neg\neg\alpha)$ Axiom 1.

$\{\neg\neg\alpha\} \vdash$ 1. $\neg\alpha \to \neg\neg\alpha$ (D.T.).

2. $\neg\alpha \to \neg\alpha$ (Theorem)

3. $(\neg\alpha \to \neg\alpha) \to ((\neg\alpha \to \neg\neg\alpha) \to \alpha)$ (Axiom 3)

4. $(\neg\alpha \to \neg\neg\alpha) \to \alpha$ (M.P. 2,3)

5. α (M.P. 1,4).

So, $\{\neg\neg\alpha\} \vdash \alpha$. Hence, $\vdash \neg\neg\alpha \to \alpha$ (D.T)

H.W. 1. $\vdash_{CPL} \alpha \to \neg\neg\alpha$

2. $\vdash_{CPL} \alpha \to (\alpha \vee \beta)$

3. $\vdash_{CPL} \alpha \to (\beta \to (\alpha \wedge \beta))$.

Is there a more structured way to do proofs?

- We can consider the formula that we want to prove and let the main connective decide how to go along with the proof.
- We can also make assumptions as and when we need, but get rid of them using implications.
- These considerations lead to the natural deduction proof system.

Natural deduction

Axiom: $\alpha \to \alpha$

Rules: - For every connective, we have two rules: introduction and elimination.

1(a) Implication introduction

$$\dfrac{\begin{array}{c}[\alpha] \\ \vdots \\ \beta\end{array}}{\alpha \to \beta} \quad \to I \text{ discharging } \alpha.$$

1 (b) Implication elimination

$$\frac{\alpha \quad \alpha \to \beta}{\beta} \qquad \to E$$

2. (a) Conjunction introduction

$$\frac{\Gamma \vdash \alpha \quad \Gamma \vdash \beta}{\Gamma \vdash \alpha \wedge \beta} \qquad \wedge I$$

2. (b) Conjunction elimination

$$\frac{\Gamma \vdash \alpha_1 \wedge \alpha_2}{\Gamma \vdash \alpha_i} \qquad \wedge E$$

3. (a) Disjunction introduction

$$\frac{\Gamma \vdash \alpha_i}{\Gamma \vdash \alpha_1 \vee \alpha_2} \qquad \vee I$$

3. (b) Disjunction elimination

$$\frac{\Gamma \vdash \alpha \vee \beta \quad \Gamma, \alpha \vdash \delta \quad \Gamma, \beta \vdash \delta}{\Gamma \vdash \delta} \qquad \vee E$$

4. (a) Negation introduction ($\perp$: false/contradiction)

$$\frac{\alpha \quad \neg \alpha}{\perp} \qquad \neg I$$

4 (b) Negation elimination

$$\dfrac{\begin{matrix}[\neg\alpha] \\ \vdots \\ \perp\end{matrix}}{\alpha} \quad \neg E$$

- We can show that the natural deduction rules are all derivable in the Hilbert-style axiom system we discussed above.

- Gentzen first introduced ND proof system to prove 'consistency of arithmetic'.
- But he was unsuccessful and introduced one more proof system : Sequent Calculus and used it to finish the proof above.

Sequent Calculus.

Sequents : $\alpha_1, \ldots, \alpha_m \vdash \beta_1, \ldots, \beta_n$

[Interpretation : $\vdash (\alpha_1 \wedge \cdots \wedge \alpha_m) \to (\beta_1 \vee \cdots \vee \beta_n)$]

The SC system consists of three classes

of rules : (i) Logical , (ii) Structural , (iii) Core.

Logical rules

* We assume two propositional constants : $\top$ (truth) and $\bot$ (false) in the language.

- Truth right rule

$$\frac{}{\Gamma \vdash \top, \Delta} \quad \top R$$

- False left rule

$$\frac{}{\Gamma, \bot \vdash \Delta} \quad \bot L$$

* Conjunction

- Right rule

$$\frac{\Gamma \vdash \alpha, \Delta \qquad \Gamma \vdash \beta, \Delta}{\Gamma \vdash \alpha \wedge \beta, \Delta} \quad \wedge R.$$

- Left rule 1

$$\frac{\Gamma, \alpha \vdash \Delta}{\Gamma, \alpha \wedge \beta \vdash \Delta} \quad \wedge L_1$$

- Left rule 2

$$\frac{\Gamma, \beta \vdash \Delta}{\Gamma, \alpha \wedge \beta \vdash \Delta} \quad \wedge L_2$$

* Disjunction

- Right rule 1:

$$\frac{\Gamma \vdash \alpha, \Delta}{\Gamma \vdash \alpha \vee \beta, \Delta} \quad \vee R_1$$

- Right rule 2:

$$\frac{\Gamma \vdash \beta, \Delta}{\Gamma \vdash \alpha \vee \beta, \Delta} \quad \vee R_2$$

- Left rule:

$$\frac{\Gamma, \alpha \vdash \Delta \qquad \Gamma, \beta \vdash \Delta}{\Gamma, \alpha \vee \beta \vdash \Delta} \quad \vee L$$

* Negation

- Right rule

$$\frac{\Gamma, \alpha \vdash \Delta}{\Gamma \vdash \neg\alpha, \Delta} \quad \neg R$$

- Left rule

$$\frac{\Gamma \vdash \alpha, \Delta}{\Gamma, \neg\alpha \vdash \Delta} \quad \neg L$$

* Implication

- Right rule

$$\frac{\Gamma, \alpha \vdash \beta, \Delta}{\Gamma \vdash \alpha \to \beta, \Delta} \quad \to R$$

- Left rule

$$\frac{\Gamma \vdash \alpha, \Delta \qquad \Gamma', \beta \vdash \Delta'}{\Gamma, \Gamma', \alpha \to \beta \vdash \Delta, \Delta'} \quad \to L$$

Structural rules

* Weakening

Right rule : $$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \alpha, \Delta}$$ WR

Left rule : $$\frac{\Gamma \vdash \Delta}{\Gamma, \alpha \vdash \Delta}$$ WL

* Contraction

Right rule $$\frac{\Gamma \vdash \alpha, \Delta}{\Gamma \vdash \alpha, \alpha, \Delta}$$ CR

Left rule $$\frac{\Gamma, \alpha, \alpha \vdash \Delta}{\Gamma, \alpha \vdash \Delta}$$ CL

* Exchange.

Right rule : $$\frac{\Gamma \vdash \Delta, \alpha, \beta, \Delta'}{\Gamma \vdash \Delta, \beta, \alpha, \Delta'}$$ XR

Left rule : $$\frac{\Gamma, \alpha, \beta, \Gamma' \vdash \Delta}{\Gamma, \beta, \alpha, \Gamma' \vdash \Delta}$$ XL

Core rules

* Axiom rule $$\frac{}{\alpha \vdash \alpha}\ Ax$$

* Cut rule $$\frac{\Gamma \vdash \alpha, \Delta \qquad \Gamma', \alpha \vdash \Delta'}{\Gamma', \Gamma \vdash \Delta, \Delta'}\ CUT$$

A few examples

1. $\Phi \vdash ((\alpha \wedge \beta) \wedge \gamma) \rightarrow (\beta \wedge \alpha)$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\beta \vdash \beta}\,Ax}{\alpha \wedge \beta \vdash \beta}\,\wedge L_2}{(\alpha \wedge \beta) \wedge \gamma \vdash \beta}\,\wedge L_1 \qquad \dfrac{\dfrac{\dfrac{}{\alpha \vdash \alpha}\,Ax}{\alpha \wedge \beta \vdash \alpha}\,\wedge L_1}{(\alpha \wedge \beta) \wedge \gamma \vdash \alpha}\,\wedge L_1}{\dfrac{(\alpha \wedge \beta) \wedge \gamma \vdash \beta \wedge \alpha}{\Phi \vdash ((\alpha \wedge \beta) \wedge \gamma) \rightarrow (\beta \wedge \alpha)}\,\rightarrow R}\,\wedge R$$

2. $\Phi \vdash \alpha \vee \neg \alpha$

$$\dfrac{\dfrac{\dfrac{}{\alpha \vdash \alpha}\,Ax}{\vdash \neg \alpha, \alpha}\,\neg R}{}\,\vee R_2$$

$$\dfrac{\dfrac{\dfrac{\vdash \alpha \vee \neg\alpha, \alpha}{\vdash \alpha, \alpha \vee \neg\alpha}\,XR}{\vdash \alpha \vee \neg\alpha, \alpha \vee \neg\alpha}\,\vee R_1}{\vdash \alpha \vee \neg\alpha}\,CR$$

3. If $\Gamma \vdash \alpha \to \beta, \Delta$ then $\Gamma, \alpha \vdash \beta, \Delta$ (D.T.)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \alpha \to \beta, \Delta}\,As}{\Gamma, \alpha \vdash \alpha \to \beta, \Delta}\,WL}{\Gamma, \alpha \vdash \alpha \to \beta, \beta, \Delta}\,WR \qquad \dfrac{\dfrac{}{\Gamma, \alpha \vdash \alpha, \beta, \Delta}\,Ax \qquad \dfrac{}{\Gamma, \alpha, \beta \vdash \beta, \Delta}\,Ax}{\Gamma, \alpha, \alpha \to \beta \vdash \beta, \Delta}\,\to L}{\Gamma, \alpha \vdash \beta, \Delta}\,CUT$$

H.W. Analyze the proof above on your own.

H.W. Using ND and SC, prove the following.

1. $\vdash_{CPL} \alpha \to \neg\neg\alpha$
2. $\vdash_{CPL} \alpha \to (\alpha \vee \beta)$
3. $\vdash_{CPL} \alpha \to (\beta \to (\alpha \wedge \beta))$.